A geometric approach to black hole spectral theory

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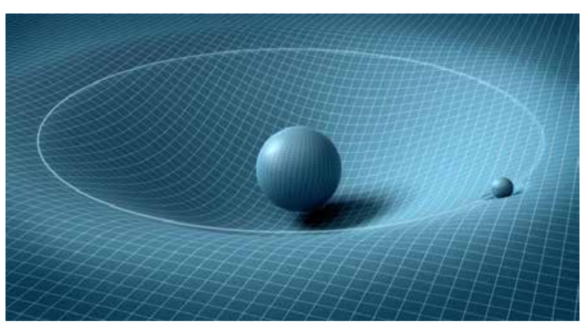
Introduction & Motivation

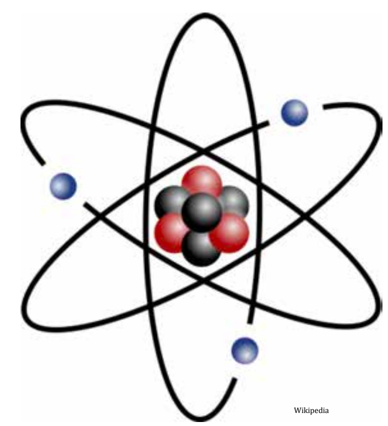
An important open problem in theoretical physics is to unify

General Relativity



resonance.is

Quantum Mechanics



Introduction & Motivation

String theory:

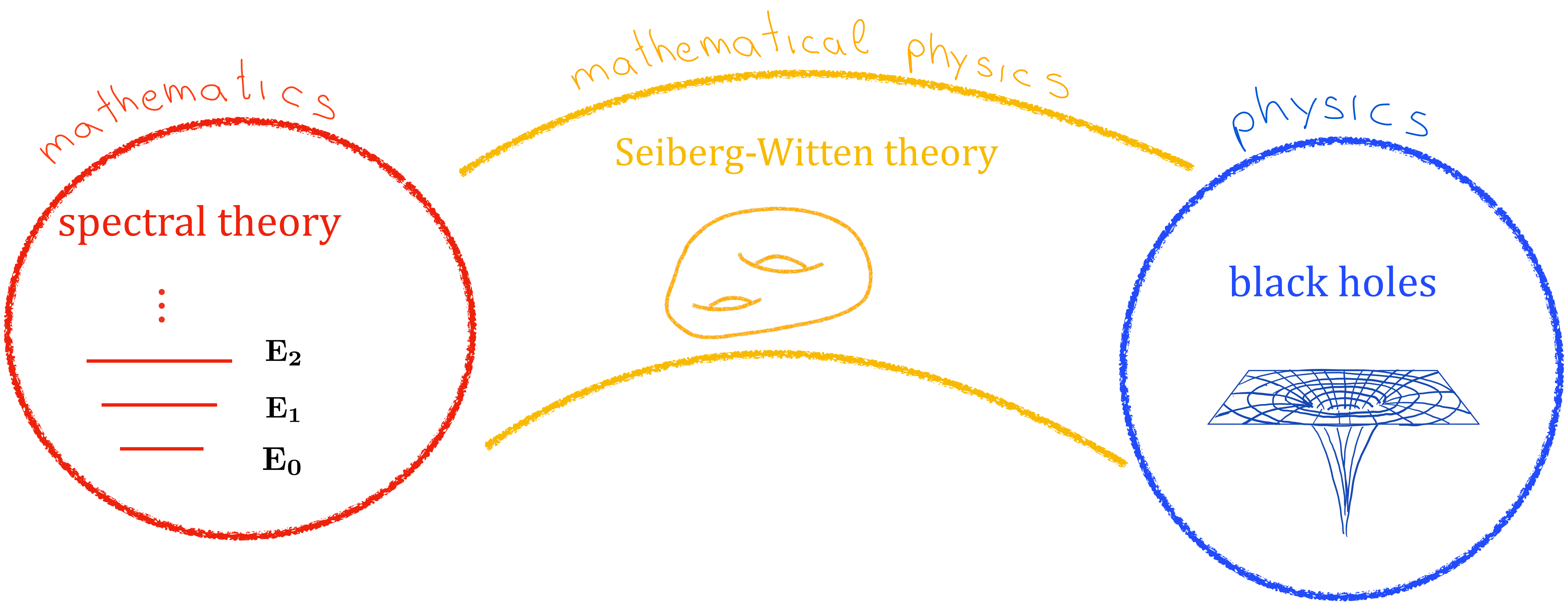
→ provides a theoretical framework to address this question

→ has generated several tools and ideas leading to new results and applications in various fields

↳ Today we will see a concrete example

Introduction & Motivation

Today's subjects



Introduction & Motivation

Exact analytic solutions for spectral theory of QM operators are rare

→ need non-perturbative tools

Fruitful guideline: think of QM geometrically [Balian-Parisi-Voros]

→ make contact with supersymmetric gauge theory and topological string

→ new non-perturbative tools

[Nekrasov-Shatashvili, Gaiotto- Moore-Neitzke, AG-Hatsuda-Marino, Huang, …]

Introduction & Motivation

Why do we expect supersymmetric gauge theory to play any role in spectral theory of QM operators?

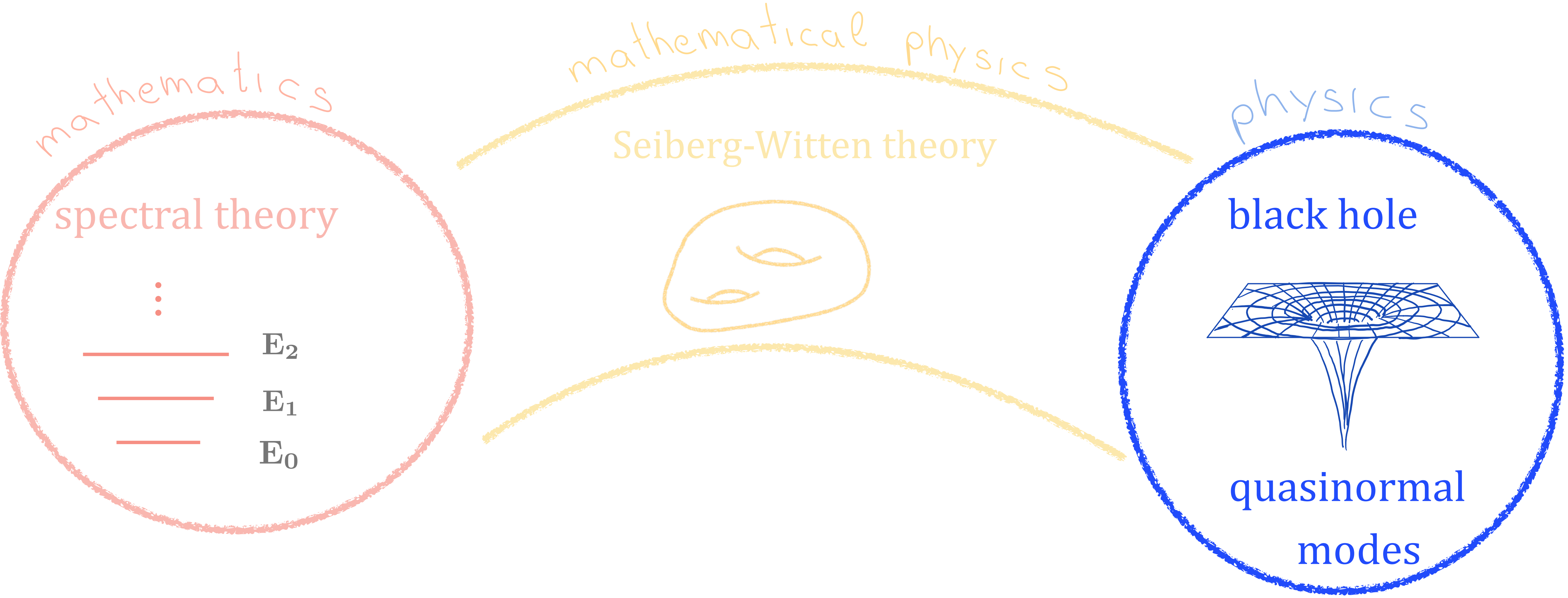
1. Balian, Parisi and Voros developed in the ‘70s a geometrical approach to quantum mechanics

2. Seiberg and Witten developed in the ‘90s a geometrical approach to supersymmetric gauge theories

→ geometry is the common background connecting these two subjects

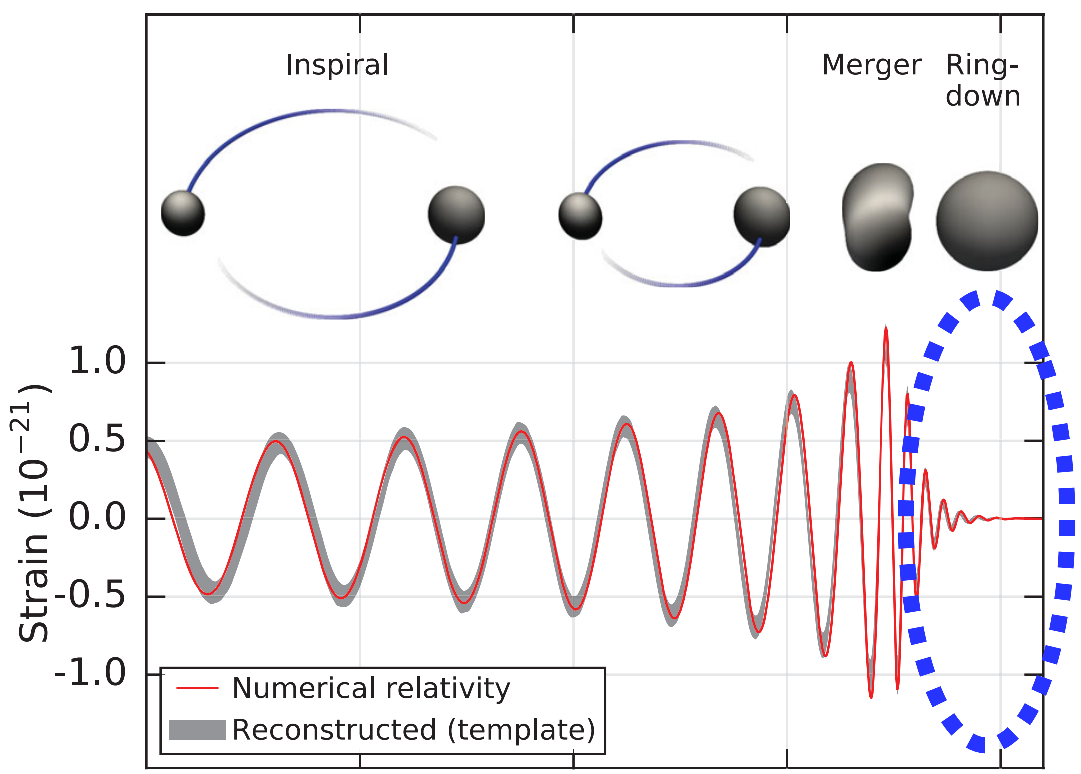
Introduction & Motivation

Today's subjects



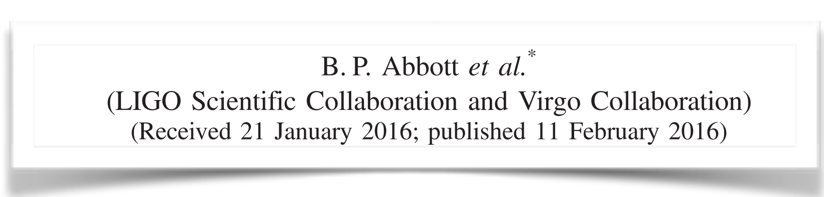
Introduction & Motivation

Black hole quasinormal modes $\{\omega_n\}_{n\geq 0}$ (QNMs) $\sim$ resonances (or dissipative modes) encoding the response of the BH to a perturbation.



damped oscillations

$$\sim \sum_n A_n \exp[-i\omega_n t], \quad \mathrm{Im}(\omega_n) < 0$$

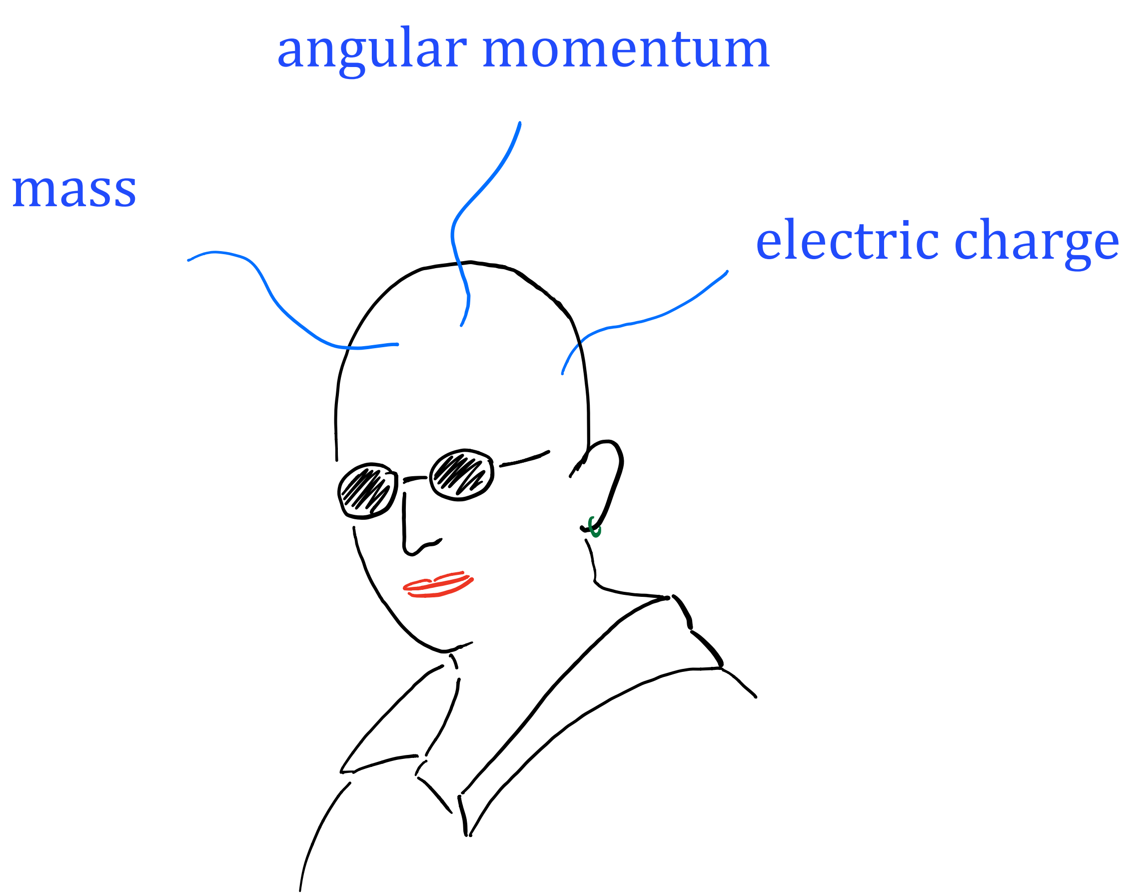


B. P. Abbott *et al.*[*]
(LIGO Scientific Collaboration and Virgo Collaboration)
(Received 21 January 2016; published 11 February 2016)

Introduction & Motivation

QNMs can be used to determine mass, angular momentum (and electric charge) of the final black hole.

Indeed, according to general relativity "black holes have no-hair" (only 3 hairs):



Introduction & Motivation

QNMs depend only on mass, angular momentum (and electric charge) of the final black hole

→ are used to determine mass, angular momentum (and electric charge) of the final black hole

→ Testing the No-Hair Theorem with GW150914

Maximiliano Isi, Matthew Giesler, Will M. Farr, Mark A. Scheel, and Saul A. Teukolsky
Phys. Rev. Lett. **123**, 111102 – Published 12 September 2019

Plan:

Review the main ideas behind the geometric/gauge theoretic approach to spectral theory and show a concrete application to the study of black hole quasinormal modes.

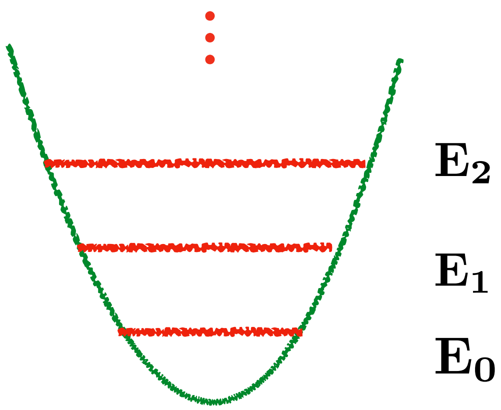
. . . spectral theory of QM operators. . .

A toy model: the harmonic oscillator

Hamiltonian: $H(\hat{x},\hat{p}) = \hat{p}^2 + \hat{x}^2, \qquad [\hat{x},\hat{p}] = i\hbar$

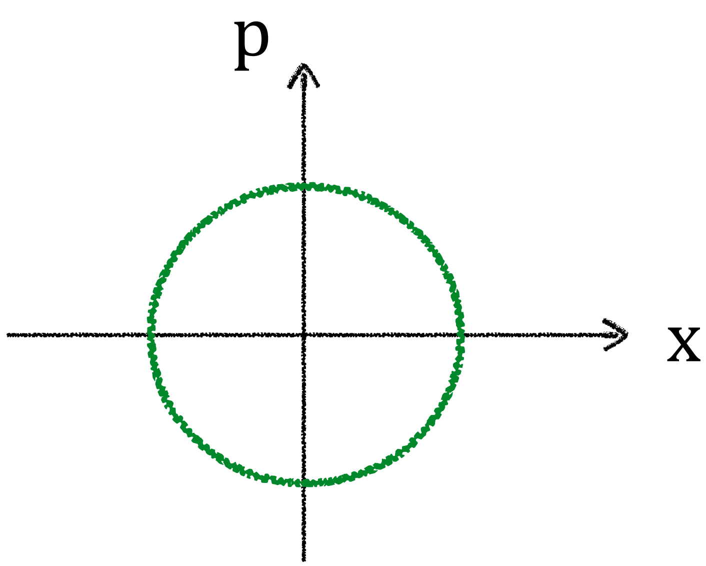
Schrödinger equation: $(-\hbar^2\partial_x^2 + x^2 - E)\phi(x) = 0$

$\longrightarrow$ positive discrete spectrum
$\{E_n\}_{n\geq 0}$ on $L^2(\mathbb{R})$



The eigenvalues can be obtained by using Bohr-Sommerfeld quantization condition:

→ quantization of semiclassical phase space volume



$$\mathrm{Vol}(E) = 2\pi\hbar\left(n + \frac{1}{2}\right), \quad n = 0,1,2,\ldots$$

$$\mathrm{Vol}(E) = \left\{x^2 + p^2 \leq E\right\} = \pi E$$

$$\rightarrow \quad E_n = 2\hbar\left(n + \frac{1}{2}\right)$$

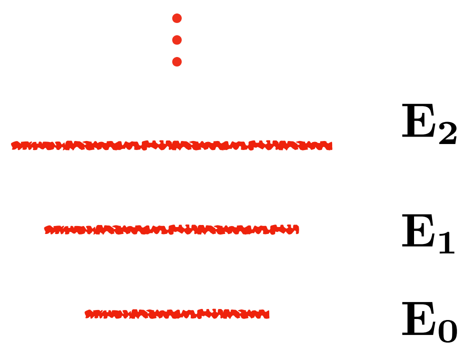
! For more generic spectral problems, Bohr-Sommerfeld is only a semiclassical approximation.

We need to include $\hbar$ corrections both at the perturbative and non-perturbative level: this is highly non trivial.

A simple example: modified Mathieu.

Hamiltonian: $$H(\hat{x},\hat{p}) = \hat{p}^2 + 2\Lambda^2 \cosh \hat{x}, \qquad [\hat{x},\hat{p}] = i\hbar$$

$\Longrightarrow$ discrete spectrum $\{E_n\}_{n\geq 0}$

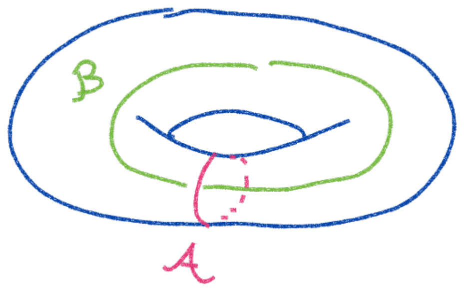


Thinking geometrically:

operator

$$H(\hat{p},\hat{x}) = \hat{p}^2 + 2\Lambda^2 \cosh \hat{x} \longrightarrow p^2 + 2\Lambda^2 \cosh x = E$$

curve



Classical periods:

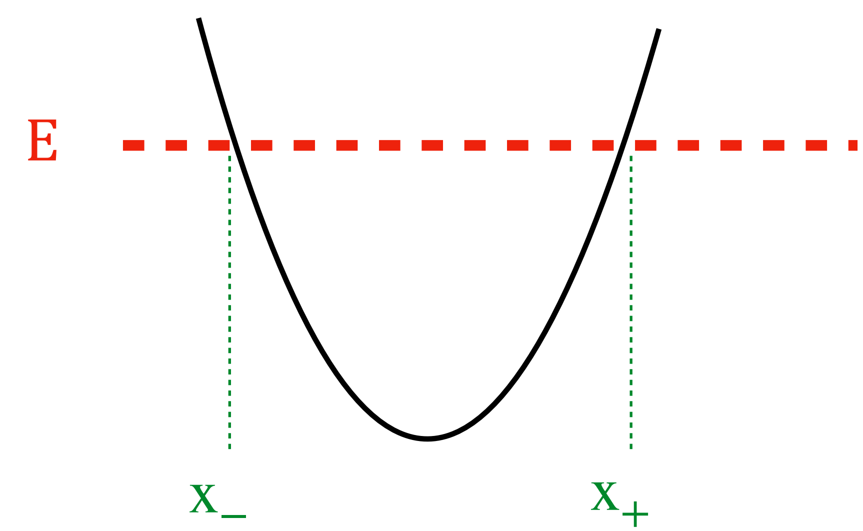
$$\Pi^{(0)}_{A,B}(E) = \oint_{A,B} p(x,E)\mathrm{dx} \quad \text{where} \quad p(x,E) = \sqrt{E - 2\Lambda^2 \cosh x}$$

- $$\Pi_B^{(0)}(E) = \int_{x_-}^{x_+} p(x,E)\mathrm{d}x = 8\sqrt{E+2\Lambda^2}\left[\mathbf{K}\left(\frac{E-2\Lambda^2}{E+2\Lambda^2}\right) - \mathbf{E}\left(\frac{E-2\Lambda^2}{E+2\Lambda^2}\right)\right]$$

$\sim$ classical trajectory

$$\mathrm{x}_\pm = \pm\cosh^{-1}\left(\frac{E}{2\Lambda^2}\right)$$

$$V(x) = 2\Lambda^2\cosh x$$



- $$\Pi_A^{(0)}(E) = \frac{1}{2\pi i}\int_{-i\pi}^{i\pi} p(x,E)\mathrm{d}x = = 8\sqrt{E+2\Lambda^2}\mathbf{E}\left(\frac{4\Lambda^2}{2\Lambda^2+E}\right)$$

$\sim$ complex trajectory

The quantization condition for this operator takes the following form:

$$\underbrace{\oint_B p(x,E)\mathrm{dx}}_{\Pi_B^{(0)}(E)} + \mathcal{O}(\hbar^2) + \underbrace{\mathcal{O}(\mathrm{e}^{-1/\hbar})}_{\text{non-pert.}} = 2\pi\hbar\left(n+\frac{1}{2}\right)$$

Perturbative corrections:

(all-order) WKB approach

[Dunham]

Terminology: we refer to l.h.s. as quantum B period: $\Pi_B(E,\hbar)$

Interesting result:

quantum periods $\Longleftrightarrow$ partition function of suitable four dimensional Seiberg-Witten theory in Ω background

[Nekrasov-Shatashvili, Mironov-Morozov, …]

Bethe/gauge correspondence

. . . Seiberg-Witten theory and the Ω background . . .

For modified Mathieu the relevant gauge theory is a supersymmetric version of four dimensional Yang-Mills with gauge group G= SU(2)

(4dim $\mathcal{N} = 2, SU(2)$ pure Seiberg-Witten theory in the Ω background)

Relevant parameters:

$\epsilon_1, \epsilon_2 \sim$ Ω background parameters

$\Lambda \sim$ instanton counting parameter ($\sim$ gauge coupling)

E (or a) $\sim$ parametrise moduli space of vacua

Two important results:

a The exact partition function $Z(a, \Lambda, \epsilon_1, \epsilon_2)$ was computed by Nekrasov

Notation: $F \sim \log Z$ is the free energy

The Nekrasov-Shatashvili phase is defined by

$$\epsilon_2 = 0$$

In our example, the explicit expression of the free energy reads:

$$F(a, \Lambda, \epsilon_1) = \sum_{n \geq 1} c_n(a, \epsilon_1) \Lambda^n$$

$$c_1(a, \epsilon_1) = -\frac{2}{a^2 + \epsilon_1^2}$$

$$c_2(a, \epsilon_1) = \frac{7\epsilon_1^2 - 5a^2}{(a^2 + \epsilon_1^2)^3 (a^2 + 4\epsilon_1^2)}$$

$$\cdots$$

Nekrasov free energy is exact in ϵ's and is a convergent series in Λ.

[Its et al, Bershtein et al, Felder et al,] b The physics of this type of gauge theories can be encoded geometrically in the Seiberg-Witten curve. In our example the relevant geometry is

$$p^2 + 2\Lambda^2 \cosh x = E$$

For example:

- the genus of the curve gives the rank of the gauge group
- the classical periods $\Pi^{(0)}_{A,B}(E)$ are related to the charges of (BPS) particles in the gauge theory.

. . . Back to modified Mathieu . . .

Modified Mathieu:

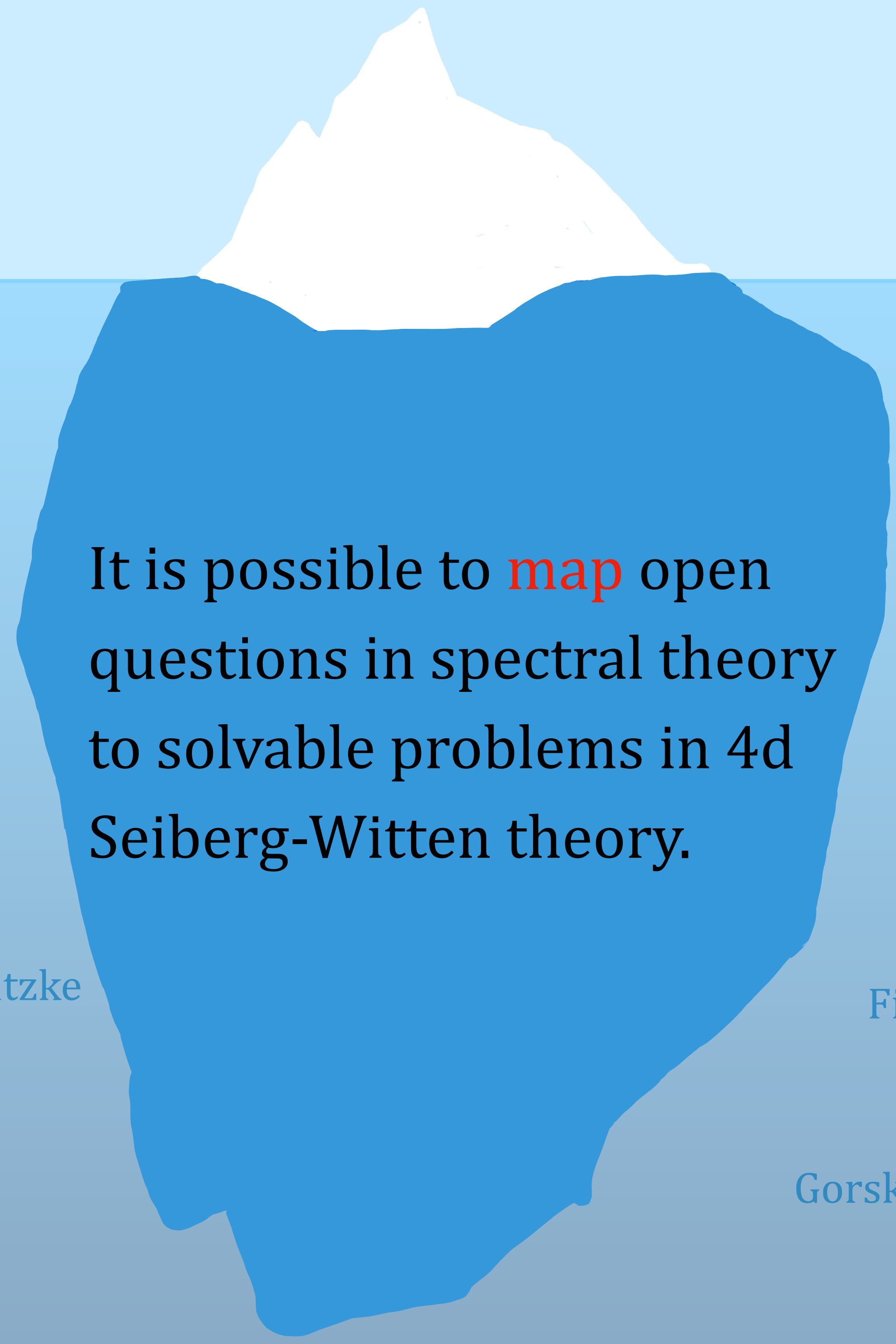
$$H(\hat{x}, \hat{p}) = \hat{p}^2 + 2\Lambda^2 \cosh \hat{x} \longrightarrow p^2 + 2\Lambda^2 \cosh x = E$$

This is the Seiberg-Witten curve of 4 dim $\mathcal{N} = 2$ SU(2) SYM

Modified Mathieu is interpreted as the quantum SW curve of $\mathcal{N} = 2$ SU(2) SYM

Moreover: $\hbar = \epsilon_1$

matching of classical curve



It is possible to map open questions in spectral theory to solvable problems in 4d Seiberg-Witten theory.

Nekrasov-Rosly-Shatashvili

Zenkevich

Mironov-Morozov

Codesido-Marino

Nekrasov-Shatashvili

Gaiotto-Moore-Neitzke

AG-Marino

He-Miao

Kozlowski-Teschner

Beccaria

.....

Many others

Kashani-Poor-Troost

Huang

AG-Gu-Marino

Ito-Shu

Gaiotto

Yan-Neitke

Hollands-Neitzke

Fioravanti-Poghossian

Dunne-Basar

Gorsky-Milekhin-Sopenko

Jeong

Gorsky, Krichever, Marshakov, Mironov Morozov,

.....

Exact quantum periods

Nekrasov-Shatashvili free energy

$$\Pi_B(E, \Lambda, \hbar)$$

$$F(a, \Lambda, \hbar)$$

$$\Pi_B(E, \Lambda, \hbar) = \frac{a}{2} \log\left(\frac{\hbar^2}{\Lambda^2}\right) - \frac{\pi\hbar}{4} - \frac{i\hbar}{2}\left(\log\Gamma\left(1 + \frac{ia}{\hbar}\right) - \log\Gamma\left(1 - \frac{ia}{\hbar}\right)\right) + \partial_a F(a, \Lambda, \hbar)$$

$$E = a^2 + \Lambda \partial_\Lambda F(a, \Lambda, \hbar) \qquad \text{(Matone relation)}$$

Therefore the exact quantization condition reads

$$\Pi_B(E,\hbar) = 2\pi\hbar\left(n+\frac{1}{2}\right) \qquad\qquad n = 0,1,2,\ldots$$

Computed exactly by using Nekrasov-Shatashvili partition function.

(proven by Kozlowski and Teschner)

Notation: $\Pi_B(E,\hbar) = \partial F^{NS}(E,\hbar)$

Partial summary

The geometric/gauge theoretic approach provides us with a new analytic window on spectral theory.

Here we focused on quantization condition, however this approach can also be used to compute eigenfunctions, Fredholm determinants, and others objects in spectral theory.

In recent years many operators of interest in mathematical physics have been successfully analysed in a similar way.

Next: apply these ideas to black hole perturbation theory and more precisely to black hole quasinormal modes

[based on work in collaboration with G. Aminov and Y. Hatsuda]

. . . back to BH quasinormal modes . . .

The framework to compute BH quasinormal modes is called BH perturbation theory.

Example:

Schwarzschild black hole: static and spherically symmetric solution to the Einstein equation in the vacuum

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

where $g_{\mu\nu}$ metric

$R_{\mu\nu}$ Ricci tensor

R scalar curvature

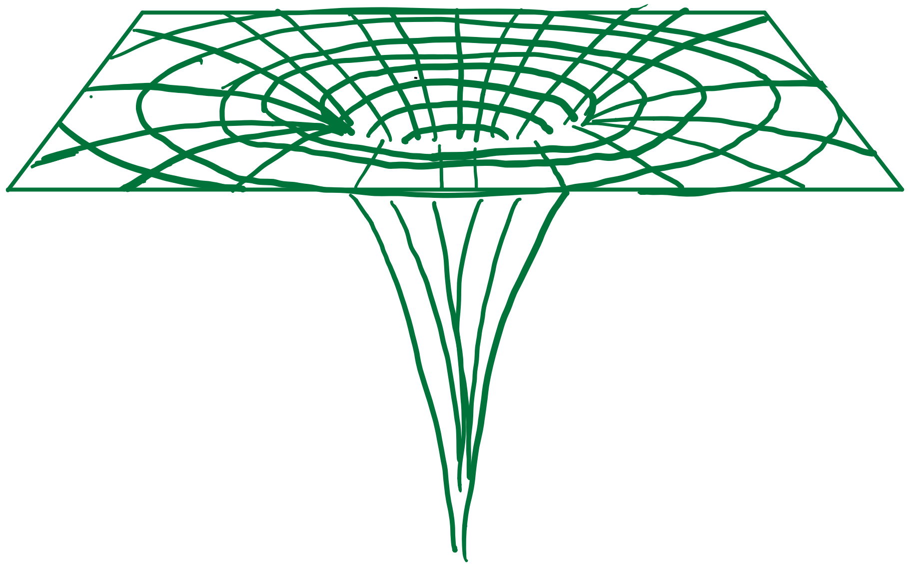
Schwarzschild metric:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

$r \to \infty$: Minkowski flat spacetime

$r = 2M$: BH horizon

$r = 0$: BH singularity



What happen if we add a "small" perturbation to this solution?

$$g_{\mu\nu} = g_{\mu\nu}^{S} + \delta g_{\mu\nu}$$

Schwarzschild metric perturbation

It was shown by Regge and Wheeler that (linear) perturbations of the Schwarzschild metric can be encoded in a simple second order differential equation.

To derive such equation, it is convenient to use the symmetries of the background metric and decompose the perturbation as:

$$\delta g = \sum_{\ell} \begin{pmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{pmatrix} \mathrm{e}^{-i\omega t} \sin\theta \frac{\partial}{\partial\theta} Y_{\ell,0}(\theta)$$

where $Y_{\ell,0}(\theta) \sim P_{\ell}(\cos\theta)$ are the spherical harmonics

Legendre polynomials

Then, substituting this into Einstein equations we obtain the Regge-Wheeler equation:

$$\left[f(r)\frac{d}{dr}f(r)\frac{d}{dr}+\omega^2-V(r)\right]\Phi(r)=0, \qquad f(r)=1-\frac{2M}{r}$$

where

$$V(r)=f(r)\left(\frac{\ell(\ell+1)}{r^2}-\frac{6M}{r^3}\right)$$

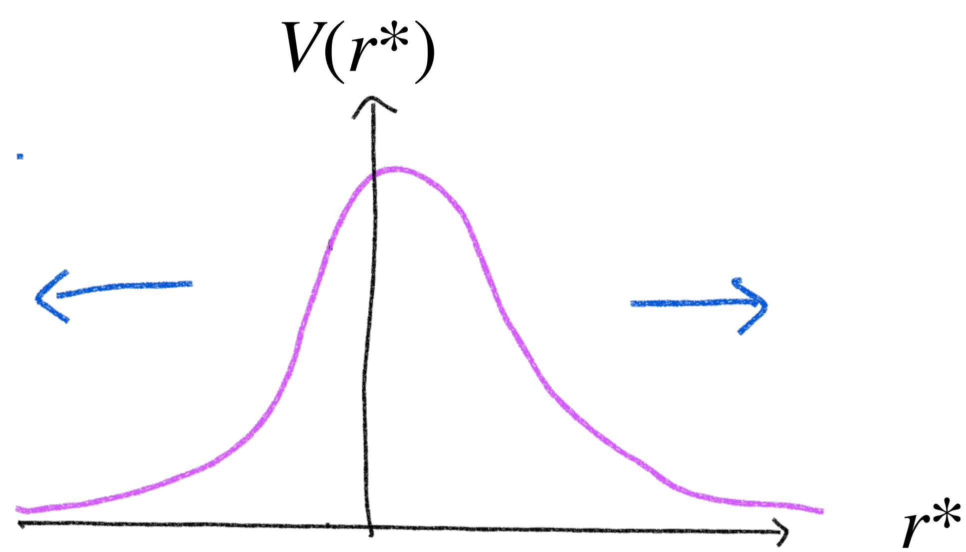
$$\left(h_1(r)\sim rf^{-1}(r)\Phi(r) \qquad h_0(r)\sim f(r)\frac{\mathrm{d}}{\mathrm{dr}}\Phi(r) \right)$$

The Regge-Wheeler equation is supplied by appropriate boundary conditions

tortoise coordinate: $r^* = 2 + 2M \log\left(\frac{r}{2M} - 1\right)$ $\longrightarrow$ horizon @ $r^* \to -\infty$

@ $r^* \to -\infty$

$\Phi(r^*) \sim \mathrm{e}^{-i\omega r^*}$



@ $r^* \to \infty$

$\Phi(r^*) \sim \mathrm{e}^{i\omega r^*}$

These BC are satisfied only for a discrete (complex) set of the frequencies $\{\omega_n\}_{n\geq 0}$ called quasinormal modes.

It is possible to compute BH quasinormal modes numerically.

[see Berti-Cardoso-Starinets (review)]

Example: $\ell = 2$ and 2M=1

https://pages.jh.edu/~eberti2/ringdown/

$$\omega_0 = 0.7473\ldots - i \cdot 0.1779\ldots$$
$$\omega_1 = 0.6934\ldots - i \cdot 0.5478\ldots$$
$$\omega_2 = 0.6021\ldots - i \cdot 0.9565\ldots$$
$$\vdots$$

Q: Is it possible to have a more analytic approach to QNMs?

We found that:

Regge-Wheeler equation $\longleftrightarrow$ Suitable quantum Seiberg-Witten geometry

some algebra

+ previous works by Zenkevich, Ito et al, Fiziev et al,

$\rightarrow$ By following the geometric/ gauge theoretic approach to spectral theory we can write an exact quantization condition for BH quasinormal modes

The relevant gauge theory is a 4 dimensional supersymmetric version of SU(2) Yang-Mills with matter (the $N_f = 3$ Seiberg-Witten theory)

The precise dictionary is:

	SYM with $N_f = 3$	Schwarzwild BH
gauge coupling	Λ	$-16iM\omega$
parametrisation of vev	E	$-\ell(\ell+1)+8M^2\omega^2-\frac{1}{4}$
flavour masses	m_1	$2-2iM\omega$
	m_2	$-2-2iM\omega$
	m_3	$-2iM\omega$
	$\hbar$	1

$$\partial F^{NS}(E, \hbar, \Lambda, m_1, m_2, m_3) = 2\pi\hbar\left(n + \frac{1}{2}\right)$$

Nekrasov-Shatashvili free energy for the $N_f = 3$ SW theory evaluated @

$$\Lambda = -16iM\omega \qquad E = -\ell(\ell+1) + 8M^2\omega^2 - \frac{1}{4} \qquad \hbar = 1$$

$$m_1 = 2 - 2iM\omega, \quad m_2 = -2 - 2iM\omega, \quad m_3 = -2iM\omega$$

$\longrightarrow$ This agrees with numerical calculations of ω_n

- The gauge theoretic approach to spectral theory can be used to obtain new analytic results for QNMs of Schwarzschild black hole

- This approach can also be generalised to other BHs

So far we studied Schwarzschild and Kerr solutions which are

1. Asymptotically flat at infinity

2. Four dimensional

We found that their quasinormal modes frequencies are encoded in four dimensional SU(2) SW theory with $N_f = 3$.

What happens if we modify the conditions 1 and 2 ? does the connection with SW theory still holds?

Some preliminary results indicated that this is the case. For instance asymptotically (A)dS BH in 4d are mapped to 4d SU(2) with $N_f = 4$.

Conclusion

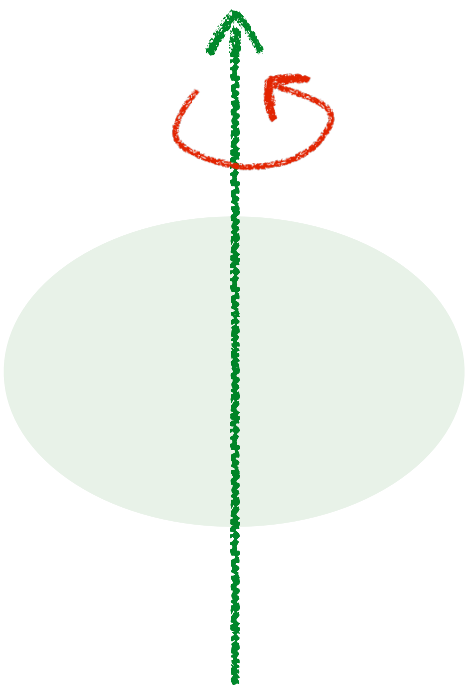
The geometric/gauge theoretic approach to spectral theory provides us with interesting non-perturbative tools which can be used to obtain new exact analytic results.

This approach has found a wide range of applications, going form integrable systems to black hole physics, which we just start to explore.

Thank you!

Kerr black hole: stationary and axially symmetric solutions to the Einstein equation in the vacuum.

They are characterised by a mass M and angular momentum α.



For the black hole horizon to exist:

$$\alpha \leq M$$

angular momentum   mass

If $\alpha = M$ the BH is called extremal.

The example of Kerr BH is technically more involved but the approach based on SW theory works and it is even more interesting:

- we obtained an analytic formula for the eigenvalues of spheroidal harmonics.

- we could easily analyse the extremal limit. This limit corresponds to the decoupling limit in SW theory where: $N_f = 3 \;\rightarrow\; N_f = 2$

The four-dimensional asymptotically flat solution in the Boyer-Lindquist coordinates is:

$$
\begin{aligned}
\mathrm{d}s^2 =& -\mathrm{d}t^2 + \mathrm{d}r^2 + 2\alpha \sin^2\theta \mathrm{d}r\mathrm{d}\phi + (r^2+\alpha^2\cos^2\theta)\mathrm{d}\theta^2 + (r^2+\alpha^2)\sin^2\theta \mathrm{d}\phi^2 \\
&+ \frac{2Mr}{r^2+\alpha^2\cos^2\theta}\left(\mathrm{d}t + \mathrm{d}r + \alpha\sin^2\theta\mathrm{d}\phi\right)^2 \quad ,
\end{aligned}
$$

$$
\left[\frac{d}{dx}(1-x^2)\frac{d}{dx} + (cx)^2 - 2csx + {}_sA_{\ell m} + s - \frac{(m+sx)^2}{1-x^2}\right]{}_sS_{lm}(x) = 0,
$$

$$x = \cos\theta$$

Dictionary for the (radial) Teukolsky equation:

	SYM with $N_f = 3$	Kerr BH
gauge coupling:	Λ	$-16i\omega\sqrt{M^2-\alpha^2}$
vev of scalar:	E	$-{}_2A_{\ell m}(\omega\alpha) - 2(2+1) + (8M^2-\alpha^2)\omega^2 - \frac{1}{4}$
flavour masses:	m_1	$2 - 2iM\omega$
	m_2	$-2 - 2iM\omega$
	m_3	$\frac{i(-2M^2\omega - \alpha m)}{\sqrt{M^2-\alpha^2}}$
	$\Lambda\partial_\Lambda F^{\mathrm{NS}}(a, m_i, \Lambda)\Big\vert_{\substack{a=\ell+1/2 \\ \Lambda=\alpha\omega \\ m_1=m \\ m_2=m_3=2}}$	$-{}_2A_{\ell m}(\omega\alpha)$: eigenvalues of spheroidal harmonics.

No closed form expression was known.