The Boltzmann Equation

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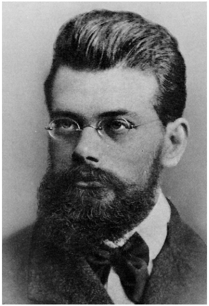
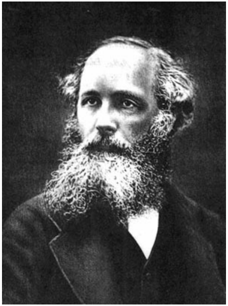
The Boltzmann Equation

Boltzmann (1872) and Maxwell (1867)

attempt at a realistic description of rarefied gases





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Boltzmann equation

$$(\partial_t + v \cdot \nabla_x) f(t,x,v) = Q(f,f)(t,x,v)$$

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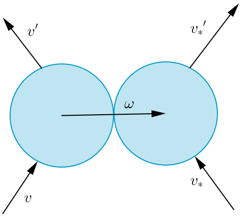
$$Q(f,f) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\omega \, B(\omega, v - v_*) \times \{ f(t,x,v'_*) f(t,x,v') - f(t,x,v_*) f(t,x,v) \}$$



Conservation laws and H-Theorem

- **Mass, Momentum, Energy**

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \varphi(v)\, f(t,x,v)\, dx\, dv = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \varphi(v)\, f_0(x,v)\, dx\, dv$$

f solution to the Boltzmann eq. with initial datum f_0 and $\varphi(v) = 1,\ v_i,\ v^2$.

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Theorem (H Theorem, Boltzmann '72)

If $f(t)$ is a regular enough solution to the Boltzmann equation, then

$$H(t) \leq H(0)$$

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On the one hand:

$$(\partial_t + v \cdot \nabla_x) f(t,x,v) = \int_{\mathbb{R}^3} dv_* \int_{S^2} d\omega \, B(\omega, v - v_*) \{ f(t,x,v'_*) f(t,x,v') - f(t,x,v_*) f(t,x,v) \}$$

looks like

$$\partial_t f(t) \sim f(t)^2 \quad \Longrightarrow \quad \text{only local in time!}$$

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there might be cancellations!

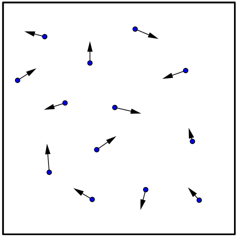
Statistical mechanics viewpoint: derivation

Classical particles

micro-scale **macro-scale**

Newton's law ($N \simeq 10^{23}$) $\Longrightarrow$ Boltzmann's equation

scaling limit



$\Longrightarrow$

effective theory

collective description

Statistical mechanics viewpoint: derivation

Newton: time **reversible** dynamics

$$\begin{cases} \dfrac{d}{dt} x_i(t) = v_i(t)\,, \\ \dfrac{d}{dt} v_i(t) = 0\,, \\ i = 1, \dots, N \end{cases} \qquad + \qquad \text{boundary conditions}$$

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Liouville equation:

$$\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N = 0 \quad + \quad \text{b.c.}$$

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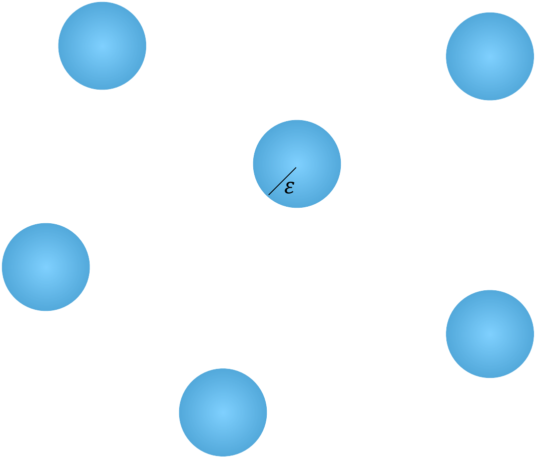
j-particle marginal:

$$f_N^{(j)}(t, x_1, v_1, \dots, x_j, v_j) = \int f_N(t, x_1, v_1, \dots, x_N, v_N)\, dx_{j+1} dv_{j+1} \dots dx_N dv_N$$

Statistical mechanics viewpoint: derivation

The Boltzmann-Grad limit: N particles of radius ε, $N \to \infty$ and $\varepsilon \to 0$



Statistical mechanics viewpoint: derivation

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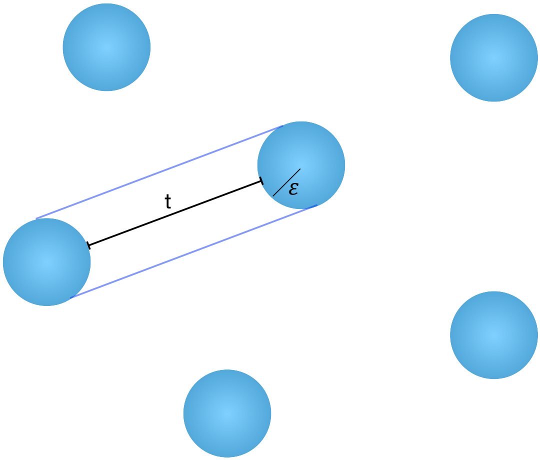
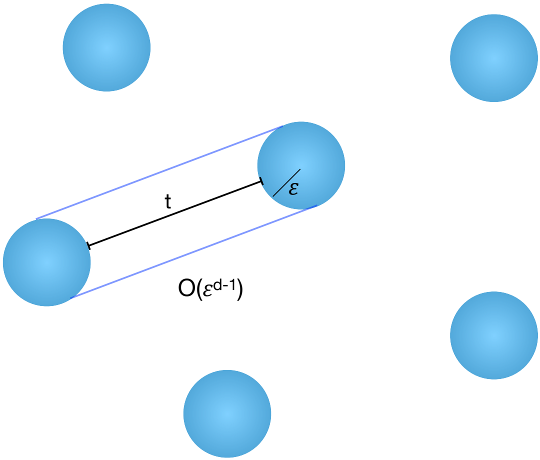


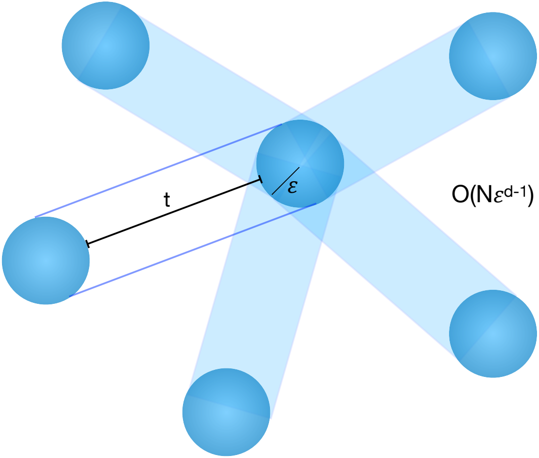
Statistical mechanics viewpoint: derivation

The Boltzmann-Grad limit: N particles of radius ε, low density regime



Statistical mechanics viewpoint: derivation

The Boltzmann-Grad limit: $N \to \infty$ with the constraint $N\varepsilon^{d-1} = O(1)$



Statistical mechanics viewpoint: derivation

$$\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N = 0 \quad + \quad \text{b.c.}$$

and consider the first marginal $f_N^{(1)}$.

Statistical mechanics viewpoint: derivation

$$(\partial_t + v\cdot\nabla_x) f_N^{(1)}(t,x,v) = (N-1)\varepsilon^2 \int_{\mathbb{R}^3}\int_{S^2} B(\omega, v-v_*) \times \{f_N^{(2)}(t,x-\varepsilon\omega, v'_*, x, v') - f_N^{(2)}(t,x+\varepsilon\omega, v_*, x, v)\}\, d\omega\, dv_*$$

to be compared with

$$(\partial_t + v\cdot\nabla_x) f(t,x,v) = \int_{\mathbb{R}^3}\int_{S^2} B(\omega, v-v_*) \times \{f(t,x,v'_*)f(t,x,v') - f(t,x,v_*)f(t,x,v)\}\, d\omega\, dv_*$$

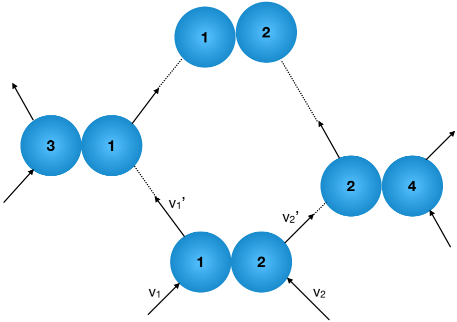
Propagation of chaos

$$f_N^{(2)}(0) \sim f_0^{\otimes 2} \quad\Longrightarrow\quad f_N^{(2)}(t) \sim f(t)^{\otimes 2}$$

where f is a solution of the Boltzmann equation with initial datum f_0.

Statistical mechanics viewpoint: derivation

Accurate study of pathological configurations



State of the art and major open problems

- Lanford (1975): hard spheres, short times
- Gallagher, Saint-Raymond, Texier (2013): quantitative analysis

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Class of interactions:

$*$ Short range potentials:
Gallagher, Saint-Raymond, Texier (2013), Pulvirenti, C.S., Simonella (2014)

$*$ Triple interactions: Ampatzoglou, Pavlovic (2019, 2020)

$*$ Long range potentials: ?

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* Long range potentials: ?

Time of validity:

* Near the vacuum: Illner, Pulvirenti (1986)
* Linear and linearized setting:
  Bodineau, Gallagher, Saint-Raymond (2016, 2017), + Simonella (2020)
* Nonlinear setting: ? (related to the global existence for the PDE)

State of the art and major open problems

...and many other open problems
(boundaries, boundary layers, molecular interactions, . . .)

Plenty of work to be done!