UNIVERSAL OPTIMALITY OF E_8 AND LEECH LATTICES

1. Introduction

The E_8 lattice and the Leech lattice are among the most famous mathematical objects. These exceptional structures emerge in dierent areas of mathematics and physics, here are only a few to name: number theory, classication of finite groups, theory of Lie groups, automorphic forms, coding theory, extremal graph theory, string theory and statistical physics. What is the reason for their "celebrity" status in nature? We cannot give a complete answer to this question. However, jointly with H. Cohn, A. Kumar, S. D. Miller, and D. Radchenko, we have recently found that these lattices have the following rare property – they are *universally optimal* [4].

2. E_8 and Leech lattices

Let us introduce the two lattices in question a little closer. Both E_8 and Leech lattice belong to the "family" of *even unimodular lattices*. We recall that a *lattice* in the Euclidean space is a discrete full rank abelian subgroups, *unimodular* lattices are the lattices containing on average 1 point per unit of volume, and the term *even* means that the length squared of each lattice vector in an even integer number. Lattices with these both properties can exist only in dimensions dividible by 8, they are rare in small dimensions and come in huge numbers in dimensions bigger then 24. The E_8-lattice is the unique even unimodular lattice in dimension 8, while the Leech lattice is one of 24 possible such lattices in dimension 24 and the only one among them having no vectors of the smallest possible non-zero length *\sqrt{2}*. Both E_8 and the Leech lattice enjoy the number of symmetries and extremal properties. For example, the shortest vectors of each of these lattices are the unique solutions to the sphere kissing problem in dimensions 8 and 24. Also these lattices are solutions to the sphere packing problem in their respective dimensions. We have found that the E_8 and the Leech lattices are extremal for a much bigger family of optimization problems.

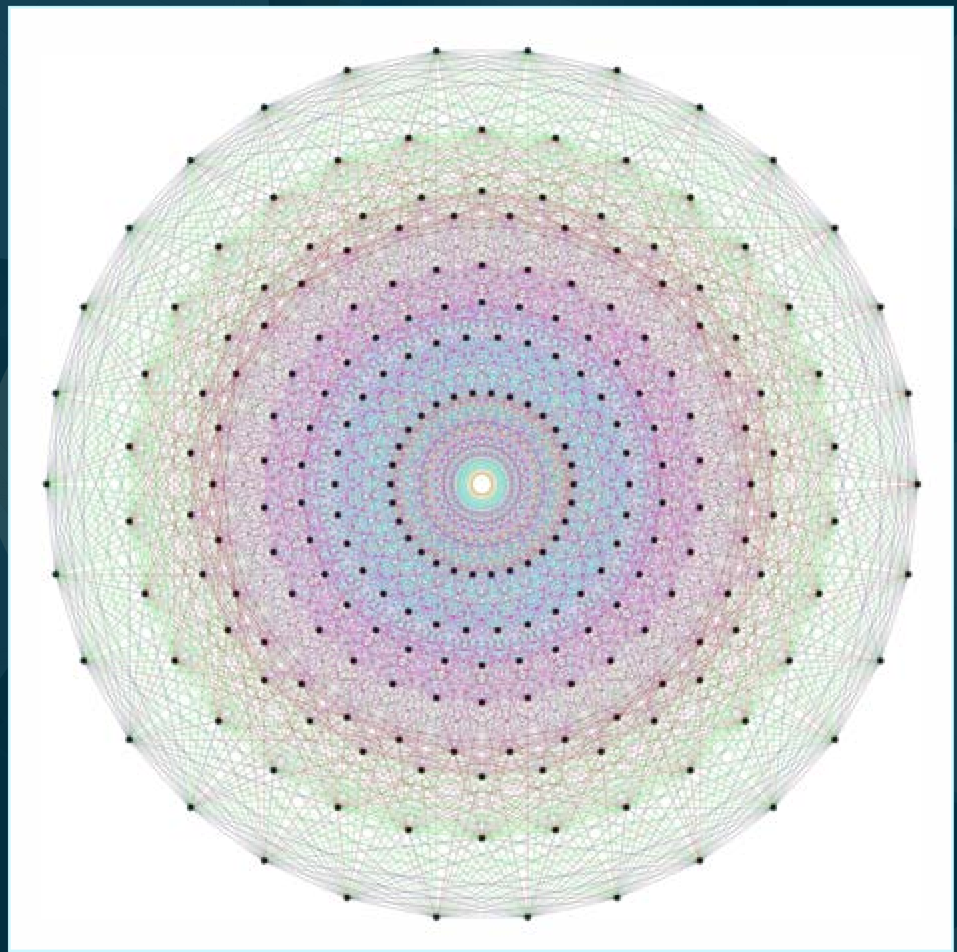

Figure 1. Petrie projection of the shortest vectors of E_8-lattice into a 2-dimensional plane. Source: Wikipedia

3. Universal optimality

Universal optimality is a property of a point conguration to be a simultaneous solution to a "universum" of optimization problems. This notion was introduced by H. Cohn and A. Kumar [2] and applied to point configurations in homogeneous spaces such as the Euclidean space, hypersphere, or hyperbolic space subject to the minimization of the energy of pairwise point interactions. We can think of points in the conguration as particles of some kind.
Energy minimization generalizes the sphere packing problem in ***R^d***, in which we wish to maximize the minimal distance between neighboring particles while fixing the particle density. In the energy minimization problem we fix the particle density *delta>0* and assume that the energy of interaction between a pair of particles depends only on distance *r* between them. Suppose that this dependence is given by a function *p(r)*. Then the *p-energy* of a conguration is the average interaction energy per particle. An energy minimization problem asks for a point conguration of density *delta* with the smallest possible value of *p-energy*. If exists, such a configuration is also called a *ground state*.

We will concentrate on energy potentials of a particular shape: we will consider repelling interactions such that the repulsion increases as particles get closer. Here is a more formal mathematical description. Recall that a function *g: (0;Infinity) ->* ***R*** is *completely monotonic* if it is infinitely differentiable and satisfies the inequalities $(-1)^k g^{(k)}$ *>= 0* for all *k >= 0*.
1. In other words, *g* is nonnegative, weakly decreasing, convex, and so on. For example, inverse power laws are completely monotonic, as are decreasing exponential functions. By Bernstein's theorem the completely monotonic functions of squared distance are the cone spanned by the Gaussians and the constant function
2. It follows that if a periodic configuration is a ground state for every Gaussian, then the same is true for every completely monotonic function of squared distance (by monotone convergence, because the potential is an increasing limit of weighted sums of finitely many Gaussians).

Following Cohn and Kumar, we call such a conguration universally optimal: Let *C* be a point configuration in ***R^d*** with density *rho*, where *rho>0*. We say *C* is *universally optimal* if WWit minimizes *p*-energy whenever *p: (0;infinity) ->* ***R*** is a completely monotonic function of squared distance.
Computations show that the universally optimal configurations are rare. A putative list of universally optimal configurations on hyperspheres is given here [2]. Examples of universally optimal configurations are also known in other metric spaces.

4. Universal optimality in Euclidean spaces

Currently, only three universally optimal configurations at the Euclidean spaces are known: the lattice of integer numbers in $\boldsymbol{R}^1$ [2], the E_8-lattice in $\boldsymbol{R}^8$, and the Leech lattice in $\boldsymbol{R}^{24}$ [4]. The hexagonal "honeycomb" lattice in dimension 2 is also conjectured to be universally optimal. A classical result is that the hexagonal lattice solves the sphere packing problem [7], however its optimality for potential energy minimization still remains open. On the other hand, the computations suggest that there is no universally optimal configuration in the 3-dimensional Euclidean space and the ground states depend on the energy profile [5], [6]. A similar situation has been detected in other small dimensions. This leads to an assumption that the existence of a universally optimal configuration in Euclidean space is an exceptional coincidence and raises a question whether it occurs only in dimensions 1, 2, 8 and 24.

Figure 2. Sphere packing in dimension 3

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Author: Maryna Viazovska
Professor, EPFL