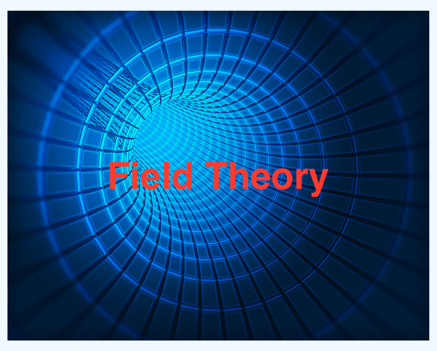

BV-BFV DESCRIPTION OF GENERAL RELATIVITY IN THREE DIMENSIONS

GIOVANNI CANEPA
UNIVERSITY OF ZURICH

SwissMAP
The Mathematics of Physics
National Centre of Competence in Research
FNSNF
SWISS NATIONAL SCIENCE FOUNDATION

ABSTRACT

The BV-BFV (Batalin-(Fradkin)-Vilkovisky) formalism is a tool to encode symmetries in gauge theories on manifolds with boundary. In this work, we compute the extension of the BV theory for three-dimensional General Relativity to all higher-codimension strata - boundaries, corners and vertices - in the BV-BFV framework. Moreover, we show that such extension is strongly equivalent to (nondegenerate) BF theory at all codimensions. This is a joint work with Michele Schiavina [CS19].

WHY BV?

- In order to make sense of path integrals using the **stationary phase formula for oscillatory integrals**, critical point of the action must be isolated. In presence of symmetries this condition is not satisfied. The BV idea is then to enlarge the space of fields to include ghost fields representing symmetries to a **Graded (-1)-symplectic manifold** and require that the extended action satisfies the **Classical Master Equation** (CME) i.e. $(S, S) = 0$ where the parenthesis are the Poisson brackets induced by the symplectic structure [BV81].
- **BV-BFV Formalism** is an extension of BV formalism on manifolds with boundary. This is useful to implement **locality** in field theory using cut and glue techniques [CMR14]. It can be extended to manifolds with corner and vertices.
- We study here the **extended BV-BFV structure of 3d Gravity** at a classical level.
- 3d General Relativity is **classically equivalent** to nondegenerate BF theory [Wit89]. The result below shows that this equivalence is preserved at a BV level [CSS18] and also as a BV-BFV theory at all codimensions.

BV-BFV N-EXTENDED THEORY

Definition 1 *Let M be an m-dim. smooth manifold. A n-**stratification** of M is a filtration $\{M^{(k)}\}_{k=0\dots n}$ such that $M^{(k)} \setminus M^{(k+1)}$ is an $(m-k)$-dim. smooth manifold.*

Remark 2 *A particular example of a stratification is given by a manifold with corners (and vertices, i.e. boundaries of corners), where the connected components of boundaries, corners and vertices compose the cells of a stratum $M^{(k)}$.*

Definition 3 *A n-**extended exact BV-BFV theory** is the assignment, to a n-stratification $\{M^{(k)}\}_{k=0\dots n}$ $(m \geq n)$ of the data*

$$\mathfrak{F}^{\uparrow n} = (\mathcal{F}^{(k)}, S^{(k)}, \alpha^{(k)}, Q^{(k)}, \pi^{(k)})_{k=0\dots n}$$

such that for every $k \leq n$:

1. *$\mathcal{F}^{(k)}$ is a graded manifold equipped with an exact symplectic form $\varpi^{(k)} = \delta\alpha^{(k)}$ with degree-$(k-1)$ where δ is the De Rham differential on $\mathcal{F}^{(k)}$;*
2. *$\pi^{(k)} : \mathcal{F}^{(k-1)} \to \mathcal{F}^{(k)}$ is a degree-0 surjective submersion;*
3. *$S^{(k)}$ is a degree-k action functional on $\mathcal{F}^{(k)}$;*

such that, for $0 \leq k \leq n-1$,

$$\iota_{Q^{(k)}}\varpi^{(k)} = \delta S^{(k)} + \pi^{(k)*}\alpha^{(k+1)} \tag{1}$$

$$\iota_{Q^{(k)}}\iota_{Q^{(k)}}\varpi^{(k)} = 2\pi^{(k)*}S^{(k+1)}, \tag{CME}$$

whereas for $k = n$, we require

$$\iota_{Q^{(n)}}\varpi^{(n)} = \delta S^{(n)}, \qquad \iota_{Q^{(n)}}\iota_{Q^{(n)}}\varpi^{(n)} = 0.$$

When $n = m$ we say that the theory is fully extended. When $n = 0$, the data is that of a BV theory.

Definition 4 *Let M be an m-dimensional smooth manifold and let $\mathfrak{F}^{\uparrow 0}$ an exact BV theory on it. We say that the BV-theory $\mathfrak{F}^{\uparrow 0}$ is n-**extendable** if, for every n-stratification such that $M^{(0)} = M$, there exist an n-extended exact BV-BFV theory $\mathfrak{F}^{\uparrow n}$ associated to it. If $n = m$ we will say that $\mathfrak{F}^{\uparrow 0}$ is **fully extendable**.*

Definition 5 *A **strong equivalence** between two n-extended exact BV-BFV theories $\mathfrak{F}_1^{\uparrow n}$ and $\mathfrak{F}_2^{\uparrow n}$ is a collection of degree-0 symplectomorphisms*

$$\Phi^{(k)} : (\mathcal{F}_1^{(k)}, \varpi_1^{(k)}) \to (\mathcal{F}_2^{(k)}, \varpi_2^{(k)})$$

preserving the k^{th} BFV action: $\Phi^* S_2^{(k)} = S_1^{(k)}$ and satisfying, for $0 \leq k \leq n-1$

$$\pi_2^{(k)} \circ \Phi^{(k)} = \Phi^{(k+1)} \circ \pi_1^{(k)}.$$

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3D-GRAVITY BV ACTION

Let $P \to M$ be an $SO(2,1)$-principal bundle on a 3-dimensional, compact, orientable smooth manifold M. Let also $\mathcal{V}$ be the associated vector bundle where each fibre is isomorphic to (V, η), a 3d vector space with a pseudo-Riemannian inner product η on it. We identify $\mathfrak{so}(2,1) \cong \bigwedge^2 \mathcal{V}$ using η. The fields are a **non degenerate co-frame field** $e \in \Omega^1_{nd}(M, \mathcal{V})$ and an $SO(2,1)$ **principal connection** $\omega \in \mathcal{A}_P \simeq \Omega^1(M, \bigwedge^2 \mathcal{V})$ (around the trivial connection).

Definition 6 *3d General Relativity (GR) is the pair $(\mathcal{F}^{cl}_{GR}, S^{cl}_{GR})$ where*

$$\mathcal{F}^{cl}_{GR} = \Omega^1_{nd}(M, \mathcal{V}) \oplus \Omega^1(M, \wedge^2\mathcal{V}) \qquad S^{cl}_{GR} = \mathrm{Tr}\int_M e \wedge F_\omega$$

are the space of fields and the action functional.

The classical functional is **invariant** under the action of internal gauge transformations $SO(2,1)$ $(c \in \Omega^0[1](M, \wedge^2\mathcal{V}))$ and the action of spacetime diffeomorphisms $(\xi \in \Gamma[1](TM))$.

Definition 7 *The **BV theory for 3d General Relativity** is given by the data $\mathfrak{F}^{\uparrow 0}_{GR} = (\mathcal{F}_{GR}, S_{GR}, \alpha_{GR}, Q_{GR})$ where the BV space of fields is*

$$\mathcal{F}_{GR} = T^*[-1]\left(\Omega^1_{nd}(M, \mathcal{V}) \oplus \mathcal{A}_P \oplus \Omega^0[1](M, \wedge^2\mathcal{V}) \oplus \Gamma[1]TM\right),$$

the BV one-form and action functional are

$$\alpha_{GR} = \mathrm{Tr}\int_M e\delta e^\dagger + \omega\delta\omega^\dagger + c\delta c^\dagger + \iota_\xi\delta\xi^\dagger,$$

$$S_{GR} = \mathrm{Tr}\int_M eF_\omega + e^\dagger\left(L^\omega_\xi e - [c, e]\right) + \omega^\dagger\left(\iota_\xi F_\omega - d_\omega c\right) + \frac{1}{2}c^\dagger\left(\iota_\xi\iota_\xi F_\omega - [c, c]\right) + \frac{1}{2}\iota_{[\xi,\xi]}\xi^\dagger,$$

where $L^\omega_\xi : [\iota_\xi, d_\omega]$. The vector field Q_{GR} is given by the defining property (1), *when M is closed and without boundary.*

BV-BFV REDUCTION

Theorem 8 *The BV theory $\mathfrak{F}^{\uparrow 0}_{GR} = (\mathcal{F}_{GR}, S_{GR}, \alpha_{GR}, Q_{GR})$ is **fully extendable**.*

The resulting structure on the boundary is given by

$$\varpi^{(1)}_{GR} = \mathrm{Tr}\int_{M^{(1)}} -\delta\tilde{e}\delta\tilde{\omega} + \delta\tilde{\omega}^\dagger\delta\tilde{c} - \delta\tilde{e}^\dagger\epsilon_n\delta\tilde{\xi}^n + \iota_{\delta\tilde{\xi}}\delta(\widetilde{ee}^\dagger) + \delta(\iota_{\tilde{\xi}}\tilde{\omega}^\dagger)\delta\tilde{\omega},$$

$$S^{(1)}_{GR} = \mathrm{Tr}\int_{M^{(1)}} -\iota_{\tilde{\xi}}\tilde{e}F_{\tilde{\omega}} - \epsilon_n\tilde{\xi}^n F_{\tilde{\omega}} - \tilde{c}d_{\tilde{\omega}}\tilde{e} + \frac{1}{2}[\tilde{c}, \tilde{c}]\tilde{\omega}^\dagger + \frac{1}{2}\iota_{\tilde{\xi}}\iota_{\tilde{\xi}}F_{\tilde{\omega}}\tilde{\omega}^\dagger + \frac{1}{2}\iota_{[\tilde{\xi},\tilde{\xi}]}\widetilde{ee}^\dagger$$
$$+ \tilde{c}d_{\tilde{\omega}}(\iota_{\tilde{\xi}}\tilde{\omega}^\dagger) + L^{\tilde{\omega}}_{\tilde{\xi}}(\epsilon_n\tilde{\xi}^n)\tilde{e}^\dagger - [\tilde{c}, \epsilon_n\tilde{\xi}^n]\tilde{e}^\dagger;$$

The actions on higher codimension strata are

$$S^{(2)}_{GR} = \int_{M^{(2)}} -\frac{1}{2}[\tilde{\tilde{c}}, \tilde{\tilde{c}}]\tilde{\tilde{e}} - \iota_{\tilde{\tilde{\xi}}}\tilde{\tilde{e}}d_{\tilde{\tilde{\omega}}}\tilde{\tilde{c}} - \epsilon_m\widetilde{\widetilde{\xi^m}}d_{\tilde{\tilde{\omega}}}\tilde{\tilde{c}} - \epsilon_n\widetilde{\widetilde{\xi^n}}d_{\tilde{\tilde{\omega}}}\tilde{\tilde{c}}$$

$$S^{(3)}_{GR} = \int_{M^{(3)}} \frac{1}{2}[\tilde{\tilde{\tilde{c}}}, \tilde{\tilde{\tilde{c}}}]\epsilon_a\tilde{\tilde{\tilde{\xi}}}^a + \frac{1}{2}[\tilde{\tilde{\tilde{c}}}, \tilde{\tilde{\tilde{c}}}]\epsilon_m\widetilde{\widetilde{\widetilde{\xi^m}}} + \frac{1}{2}[\tilde{\tilde{\tilde{c}}}, \tilde{\tilde{\tilde{c}}}]\epsilon_n\widetilde{\widetilde{\widetilde{\xi^n}}}$$

BF-GR EQUIVALENCE

BF theory has the same classical structure of 3d GR (6) but different symmetries given by gauge trasformations $(\chi \in \Omega^0(M, \mathfrak{so}(2,1)))$ and a *shift symmetry* $(\tau \in \Omega^0(M, \mathcal{V}))$. The BV one-form and action functional are

$$\alpha_{BF} = \mathrm{Tr}\int_M \mathcal{B} \wedge \delta\mathcal{A} \qquad S_{BF} = \mathrm{Tr}\int_M \mathcal{B} \wedge \left(d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}]\right)$$

where $\mathcal{B} = \tau + e + \omega^\dagger + \chi^\dagger \in \Omega^\bullet(M, \mathcal{V})[1-\bullet]$, $\quad \mathcal{A} = \chi + \omega + e^\dagger + \tau^\dagger \in \Omega^\bullet(M, \wedge^2\mathcal{V})[1-\bullet]$.

Theorem 9 *The fully extended BV-BFV theories $\mathfrak{F}^{\uparrow 3}_{GR}$ and $\mathfrak{F}^{\uparrow 3}_{BF_*}$ are **strongly equivalent**, i.e. the following diagram is commutative.*

$$\begin{array}{ccccccc}
\mathcal{F}_{GR} & \xrightarrow{\pi^{(1)}_{GR}} & \mathcal{F}^{(1)}_{GR} & \xrightarrow{\pi^{(2)}_{GR}} & \mathcal{F}^{(2)}_{GR} & \xrightarrow{\pi^{(3)}_{GR}} & \mathcal{F}^{(3)}_{GR} \\
\downarrow \psi & & \downarrow \psi^{(1)} & & \downarrow \psi^{(2)} & & \downarrow \psi^{(3)} \\
\mathcal{F}_{BF_*} & \xrightarrow{\pi^{(1)}_{BF_*}} & \mathcal{F}^{(1)}_{BF_*} & \xrightarrow{\pi^{(2)}_{BF_*}} & \mathcal{F}^{(2)}_{BF_*} & \xrightarrow{\pi^{(3)}_{BF_*}} & \mathcal{F}^{(3)}_{BF_*}
\end{array}$$

Explicit symplectomorphisms can be found between the spaces of fields $\mathcal{F}^{(k)}_{GR/BF_*}$ at every codimension, and they commute with the BV-BFV surjective submersion maps.

The results show how diffeomorphisms can be seen as an equivalent choice of a BV-extension of classical BF theory, and fully describe the compatibility with lower dimensional strata, completely characterising the symmetries of GR in three dimensions.