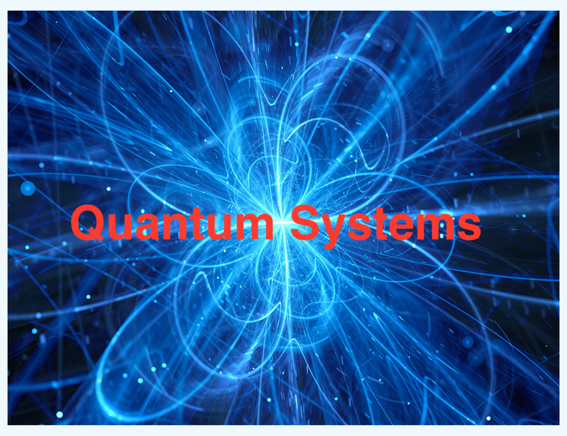

Topology in shallow-water waves

A violation of bulk-edge correspondence

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Introduction

We apply concepts used in the field of topological insulators to a classical wave phenomenon. In particular we study the correspondence between a topological invariant obtained from the bulk system defined without boundaries, with one where we introduce an edge, restricting to the half-space. Usually the two invariants agree, but we show that there is no correspondence for shallow water waves.

Shallow water equations

Equations describing a thin layer of an incompressible fluid between a flat bottom and a free surface

$$\frac{\partial\eta}{\partial t} = -\nabla\cdot\vec v\,,$$
$$\frac{\partial\vec v}{\partial t} = -g\nabla\eta - f\vec v^{\perp} - \nu\Delta\vec v^{\perp}\,.$$

Dynamical fields:

- height above the surface $\eta(x,y,t)$,
- two-component velocity field $\vec v(x,y,t)$,

($(\cdot)^{\perp}$: rotation by $\pi/2$).
Parameters: Gravity g, angular velocity $f/2$ and **odd viscosity** ν.
From $3D$ Euler equations for incompressible and homogeneous fluid.
Assumption: Typical wavelength of fluid $\gg$ height of fluid.
Regularization at small scales: $\nu\neq 0$ (allows to compactify momentum space).

Model as spin 1 (bulk)

Analogous to a Schrödinger equation $\mathrm{i}\partial_t\psi = \mathcal{H}\psi$ for $\psi=(\eta,u,v)$ with

$$\mathcal{H} = \begin{pmatrix} 0 & p_x & p_y \\ p_x & 0 & -\mathrm{i}(f-\nu\vec p^{\,2}) \\ p_y & \mathrm{i}(f-\nu\vec p^{\,2}) & 0 \end{pmatrix}.$$

Translation invariance: $\mathcal{H}$ reduces to fibers

$$H = \vec d\cdot\vec S\,,\quad \vec d(\vec k) = (k_x, k_y, f-\nu\vec k^2)\,,$$

with $\vec S$ a spin 1 irrep..
Eigenvalues separated by two gaps of size f:

$$\omega_0(\vec k)=0\,,\quad \omega_\pm(\vec k) = \pm\sqrt{\vec k^2 + (f-\nu\vec k^2)^2}\,.$$

Eigenprojections:

$$P_0 = 1-(\vec e\cdot\vec S)^2\,,\quad P_\pm = \frac12\left((\vec e\cdot\vec S)^2 \pm \vec e\cdot\vec S\right).$$

- $\vec e(\vec k)\to(0,0,-\operatorname{sign}\nu)\ (|\vec k|\to\infty)$.

Edge waves

Restrict to upper half-space $(x,y)\in\mathbb{R}\times\mathbb{R}_+$. Self-adjoint boundary condition at $y=0$:

$$v=0\,,\quad \partial_x u + a\partial_y v = 0\,,\quad a\in\mathbb{R}\,.$$

Bulk-edge correspondence?

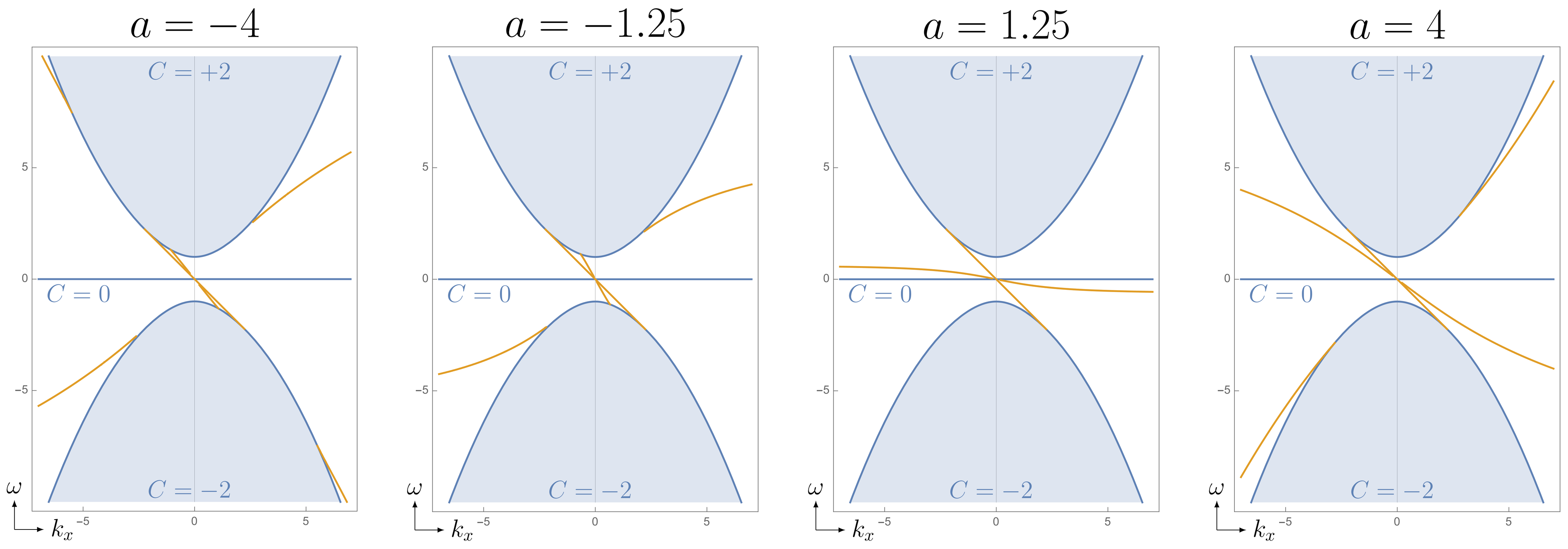
Bulk (Chern number)
Line bundles P_0, $P_\pm$ defined in terms of $\vec e(\vec k)$ on S^2 have Chern numbers

$$C(P) = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}^2} \mathrm{d}k_x\mathrm{d}k_y\ \mathrm{tr}(P[\partial_{k_x}P,\,\partial_{k_y}P])\,,$$
$$C(P_0)=0\,,\quad C(P_\pm)=\pm 2\,.$$

Edge (Hatsugai relation)
n: signed number of edge mode branches emerging (+) or disappearing (−) at the lower band limit as k_x increases.

$$C(P_+) = n = \begin{cases} 2 & (a,-\sqrt2)\,,\\ 3 & (-\sqrt2<a<0)\,,\\ 1 & (0<a<\sqrt2)\,,\\ 2 & (a>\sqrt2)\,.\end{cases}$$

→ RHS depends on a!



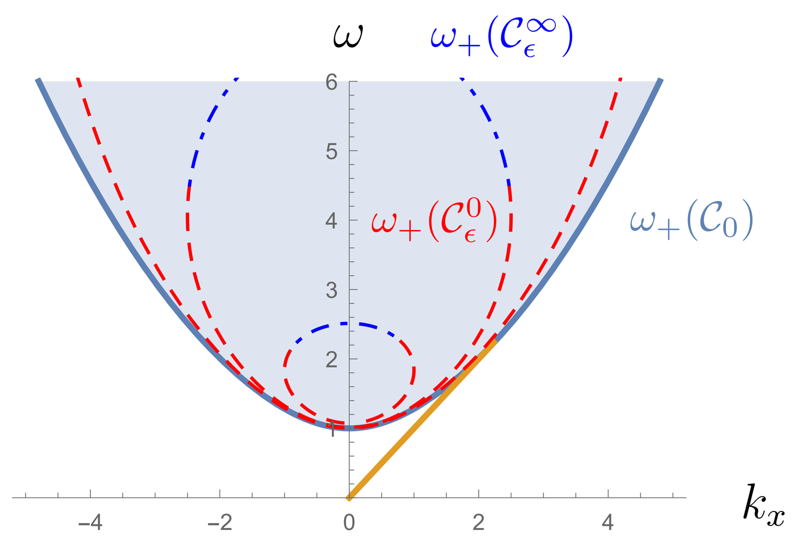
Violation of bulk-edge correspondence

Levinson's theorem: Scattering phase at energies just above thresholds ↔ bound states below it.
- $S(k_x,\omega)$: Scattering amplitude for scattering from inside the bulk: $|out\rangle = S|in\rangle$.
- $\mathcal{N}_C(f) = 1/2\pi\mathrm{i}\int_C f^{-1}\mathrm{d}f$: Winding of $f: S^1\to S^1$ along a curve C.

Parametrized Hamiltonian: $\mathcal{N}_C(S)$ ↔ number of bound states merging with threshold along C.
Theorem. Let $a\in\mathbb{R}\setminus\{0,\pm\sqrt2\}$, C_ϵ be parametrizing a loop on the sphere of momenta (see fig.), where the loop is split into parts away (C_ϵ^0) and near (C_ϵ^∞) infinite momentum.

- **Bulk-scattering correspondence** (no violation). $\forall\epsilon>0$

$$C(P_+) = \mathcal{N}_{C_\epsilon}(S)\,.$$



- **Levinson's theorem at finite momenta** (no violation).

$$n = \lim_{\epsilon\to0}\mathcal{N}_{C_\epsilon^0}(S)\,.$$

- **Levinson's theorem near infinite momentum** (violation).

$$\lim_{\epsilon\to0}\mathcal{N}_{C_\epsilon^\infty}(S) = \begin{cases} 0, & |a|>\sqrt2\,,\\ \operatorname{sign}(a), & 0<|a|<\sqrt2\,.\end{cases}$$

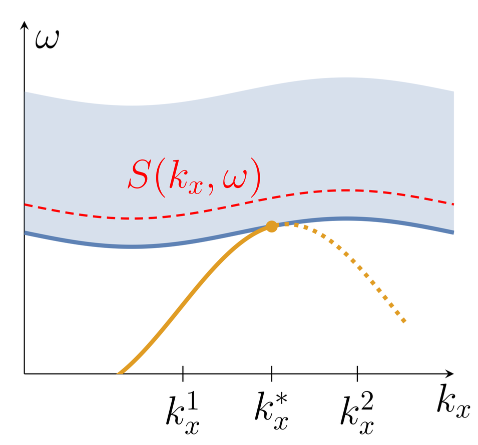
- $a \gtrless \pm\sqrt2$: Edge mode merges with the bulk band at $k_x=\pm\infty$.
- $|a|<\sqrt2$: No edge modes in neighbourhood of bulk band as $|k_x|\to\infty$.

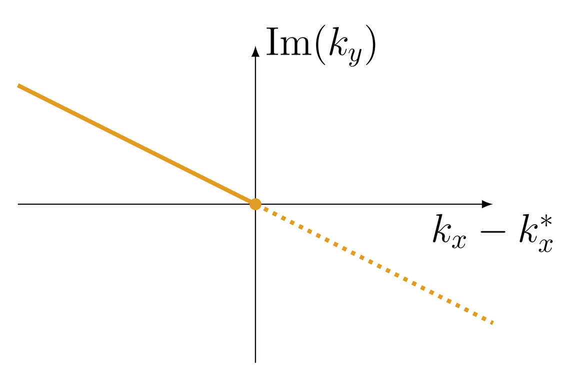
More on scattering theory

Structure of scattering amplitude:

$$S(k_x,\omega) = -\frac{g(k_x,\tilde k_y)}{g(k_x,k_y)}\,.$$

- $\tilde k_y/k_y$: incoming/outgoing momenta.
- g is analytic in k_y.
- Bound states ↔ poles of $S(k_x,\omega)$ with $\mathrm{Im}\,k_y>0$.





Future directions

- What about other boundary conditions?
- Other systems where methods are relevant?
- Geometry of complex bands "at infinity"?

Selected references

[1] Graf, G. M., and Porta, M. (2013) *Bulk-edge correspondence for two-dimensional topological insulators.* Commun. Math. Phys. **324(3)** 851-895
[2] Tauber, C., Delplace, P., and Venaille, A. (2019) *A bulk-interface correspondence for equatorial waves.* J. Fluid Mech. **868**