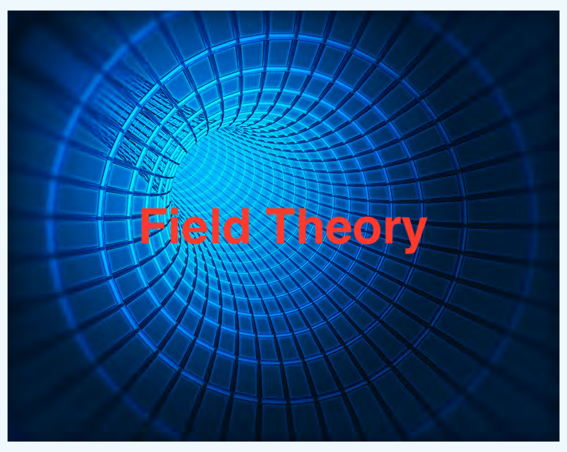

Ruelle zeta function from field theory:
A perspective on Fried's conjecture

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Abstract

We propose a field-theoretic interpretation of Ruelle zeta function, and show how it can be seen as the partition function for BF theory when an unusual gauge fixing condition on contact manifolds is imposed. This suggests an alternative rephrasing of a conjecture due to Fried on the equivalence between Ruelle zeta function and analytic torsion, in terms of homotopies of Lagrangian submanifolds. — *Based on [3].*

Ruelle zeta and Fried's conjecture

Let $M = S^*_g\Sigma$, be the unit cotangent bundle of a compact, oriented, connected, d-dimensional Riemannian manifold without boundary (Σ, g), and let $E \to M$ be a Hermitian vector bundle of rank r, equipped with a flat connection ∇ and a unitary representation $\rho : \pi_1(M) \to U(\mathbb{C}^r)$, such that the twisted de Rham complex is acyclic. Suppose that Σ has sectional curvature which is everywhere strictly negative, and denote by $\mathcal{P}$ the set of primitive orbits of the geodesic flow ϕ_t.

Definition 1 ([4]). *The Ruelle zeta function twisted by the representation ρ is*

$$\zeta_\rho(\lambda) := \prod_{\gamma\in\mathcal{P}} \det(I - \rho([\gamma])e^{-\lambda\ell(\gamma)}). \tag{1}$$

Conjecture 2 (Fried [2]). *Let (M, E, ρ) be as above. Then:*

$$|\zeta_\rho(0)|^{(-1)^{d-1}} = \tau_\rho(M) = \tau_\rho(\Sigma)^2. \tag{2}$$

where $\tau_\rho(M)$ is the Ray–Singer analytic torsion:

$$\tau_\rho(M) := \prod_{k=1}^{N} \det{}^\flat(\Delta_k)^{\frac{k}{2}(-1)^{k+1}} = \prod_{k=0}^{N-1} \det{}^\flat(d_k^* d_k)^{\frac{1}{2}(-1)^k}. \tag{3}$$

with $\Delta_k := (d^*_\nabla + d_\nabla)^2 : \Omega^k(M; E) \to \Omega^k(M; E)$ the (twisted) Laplacian on E-valued k-forms, and $\det{}^\flat$ a regularised determinant[a].

Denote by X the geodesic vector field on $S^*_g\Sigma$, and by $\Omega^\bullet_0(M)$ the space of differential forms ω such that $\iota_X\omega = 0$. We show that

Proposition 3. *The following decomposition holds*

$$\zeta_\rho(\lambda)^{(-1)^{d-1}} = \prod_{k=0}^{2n} \zeta_{\rho,k}(\lambda)^{(-1)^k} \tag{4}$$

for certain functions $\zeta_{\rho,k}(\lambda)$ such that $\det{}^\flat(\mathcal{L}_{X,k}|_{\Omega^k_0} - \lambda) = \zeta_{\rho,k}(\lambda)$. Hence

$$\zeta_\rho(0)^{(-1)^{d-1}} = \mathrm{sdet}^\flat(\mathcal{L}_X|_{\Omega^\bullet_0}). \tag{5}$$

We used here the "flat superdeterminant" $\mathrm{sdet}^\flat$, of the operator $\mathcal{L}_X$. Observe that the analytic torsion can also be seen as a regularised super determinant, by means of $\tau_\rho(M) = [\mathrm{sdet}^\flat(\Delta|_{\text{coexact}})]^{\frac{1}{2}}$

[a] Here we will systematically consider a regularisation scheme based on "flat" or "mollified" traces [1]. For Δ it coincides with the standard zeta-regularisation.

BF Theory and BV Formalism

We consider now topological BF theory on $M = S^*_g\Sigma$, i.e. the data

$$\mathcal{F}_{BF} := \Omega^{-\bullet}(M, E)[1] \oplus \Omega^{-\bullet}(M, E)[N-2] \ni (\mathbb{A}, \mathbb{B}), \tag{6}$$

together with a degree -1 symplectic form $\Omega_{BF} = \int_M [\delta\mathbb{B}\delta\mathbb{A}]^{\text{top}}$ and a degree 0 functional $\mathbb{S}_{BF} = \int_M [\mathbb{B}d_\nabla\mathbb{A}]^{\text{top}}$, such that $\{\mathbb{S}_{BF}, \mathbb{S}_{BF}\}_{\Omega_{BF}} = 0$. This defines a Batalin–Vilkovisky (BV) theory.
The starting point of quantum considerations is the partition function, formally written as the integral (we avoid discussing phases)

$$Z(\mathbb{S}_{BF}) = \int \exp(i\mathbb{S}_{BF}) \tag{7}$$

The conceptual tool to make sense of the expression (7) is the notion of gauge fixing, aimed at removing the degeneracy of the Hessian of $\mathbb{S}_{BF}$.

Fried's conjecture & gauge fixing

To compute the partition function of BF theory in the BV formalism one needs a "gauge-fixing" Lagrangian submanifold of $\mathcal{F}_{BF}$. A classical result by Schwarz [5] can be summarised as follows

Theorem 4. *Let $\mathbb{L}_g \subset \mathcal{F}_{BF}$ be the Lagrangian submanifold given by coexact forms. Then, the partition function of BF theory can be computed to be*

$$Z(\mathbb{S}_{BF}, \mathbb{L}_g) = \tau_\rho(M). \tag{8}$$

One heuristic interpretation of Schwarz's procedure is to make sense of partition functions for quadratic functionals as regularised determinants. In this case, writing $\mathbb{B} = \star\tau$, we look at the quadratic form $\mathbb{S}_{BF}|_{\mathbb{L}_g} = \sum_{k=1}^N (\tau_k, d^*_\nabla d_\nabla \mathbb{A}_k)|_{\text{coexact}}$, where $(\cdot,\cdot)$ is the inner product on k-forms induced by g. In this spirit we prove the following:

Proposition 5 ([3]). *Let X be the geodesic vector field on $M = S^*\Sigma$. Then,*

$$\mathbb{L}_X := \{(\mathbb{A}, \mathbb{B}) \in \mathcal{F}_{BF} \mid \iota_X\mathbb{B} = 0;\ \iota_X\mathbb{A} = 0\}$$

is Lagrangian in $\mathcal{F}_{BF}$. We denote this condition as contact gauge.

Theorem 6 ([3]). *The partition function of BF theory in the contact gauge is*

$$Z(\mathbb{S}_{BF}, \mathbb{L}_X) = |\zeta_\rho(0)|^{(-1)^{d-1}}. \tag{9}$$

Observe that, writing $\mathbb{B} = \iota_X\tau$, we get $\mathbb{S}|_{\mathbb{L}_X} = (\tau\mathcal{L}_X\mathbb{A})|_{\Omega^\bullet_0}$, and we are lead to the following:

Claim 7. *Proving gauge-fixing independence of the partition function of BF theory in the Batalin–Vilkovisky formalism would imply Conjecture 2.*

Homotopies and the BV theorem

The natural question now is: "how does one prove gauge fixing independence for the case at hand?"

The full BV framework controls the dependency on gauge fixing by assuming the existence of a second order operator on $C^\infty(\mathcal{F}_{BF})$, called BV Laplacian Δ_{BV}: ideally, whenever $\Delta_{BV}\exp(-\mathbb{S}_{BF}) = 0$, the partition function is constant on a family of gauge fixing Lagrangians $\mathbb{L}_t$.

For this idea to work in infinite dimensional cases like this one, we need to ensure that Δ_{BV} is appropriately defined and regularised (this is guaranteed in finite dimensions), and that there exists a homotopy of Lagrangian submanifolds $\mathbb{L}_t$ connecting $\mathbb{L}_g$ to $\mathbb{L}_X$.

This offers a new angle to tackle Fried's conjecture, replacing the microlocal analysis of Ruelle zeta function with the geometry of Lagrangian submanifolds in $\Omega^\bullet(M, E)$, and the problem of appropriately extending the BV theorem to BF theory.

On the other hand, such a bridge between field theory and modern analysis works both ways, effectively allowing us to prove gauge-fixing independence of BF theory using Fried's conjecture (true e.g. for surfaces), and to port powerful techniques in microlocal analysis to field theory, yielding a nontrivial new perspective on field theory.

References

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