DERIVED REPRESENTATION SCHEMES - WITH APPLICATIONS TO PARTITION FUNCTIONS OF QUIVER GAUGE THEORIES

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REPRESENTATION SCHEMES

For an algebra $A \in \mathtt{Alg}_k$, the scheme $\mathrm{Rep}_n(A) = \mathrm{Spec}(A_n)$ of its representations in a vector space $V = k^n$ and the character scheme $\mathrm{Rep}_n^{GL_n}(A) := \mathrm{Rep}_n(A)/GL_n$ (classifying isomorphism classes of such representations) have been studied since a long time.

Examples

1. $A = k[x_1, \ldots, x_d] \rightsquigarrow \mathrm{Rep}_n(A) = C(d, n)$ the scheme of d-tuples of $n \times n$ commuting matrices.
2. $A = k[x]/(x^d) \rightsquigarrow \mathrm{Rep}_n(A) = \mathrm{Mat}_n^{(d)}$ the scheme of matrices $X \in \mathrm{Mat}_n$ such that $X^d = 0$.
3. $A = k[Q]$ the path algebra of a quiver, and $\underline{\mathbf{n}}$ is a dimension vector for the vertices of the quiver $\rightsquigarrow \mathrm{Rep}_{\underline{n}}(A) = \mathrm{Rep}(Q, \underline{\mathbf{n}})$ the linear space of representations of the quiver, and $\mathrm{Rep}_{\underline{n}}^{GL_{\underline{n}}}(A)$ the corresponding affine quiver variety parametrising conjugacy classes of such representations.

DERIVED REPRESENTATION SCHEMES

Noncommutative geometry principle: any significant geometric structure on a noncommutative algebra A should induce the corresponding commutative structure on the representation schemes $\mathrm{Rep}_n(A)$. **Problem**. This seems to fail when the representation schemes are not smooth.

Possible solution ([1],[2]). Find a *smoothening* of $\mathrm{Rep}_n(A)$ by extending the representation functor to the model category of differential graded algebras

$$(-)_n = \mathcal{O}\mathrm{Rep}_n(-) : \mathtt{DGA}_k \to \mathtt{CDGA}_k$$

Use the total left derived functor $\mathbb{L}(-)_n$ to define the desired smoothening $\mathrm{DRep}_n(A) := \mathrm{Spec}(\mathbb{L}(A)_n)$. This is a derived affine scheme whose underlying truncation is the classical representation scheme: $\pi_0(\mathrm{DRep}_n(A)) = \mathrm{Spec}(H_0(\mathbb{L}(A)_n)) = \mathrm{Rep}_n(A)$.

The *representation homologies* $H_i(A, n) := H_i(\mathbb{L}(A)_n)$ have some important features (see [1]), and they provide numerous invariants for A:

1. They are modules over A_n, so under some finite generatedness condition they define classes $[H_i(A, n)] \in K(\mathrm{Rep}_n(A))$.
2. When everything is bounded there is a well-defined Euler characterstic as the alternate sum of these homologies: $\chi(A, n) \in K(\mathrm{Rep}_n(A))$.
3. $\mathrm{DRep}_n^{GL_n}(A) := \mathrm{DRep}_n(A)/GL_n$ yields other invariants by decomposing

$$H_i(A, n) = \bigoplus_{\lambda \in \mathrm{Irr}(GL_n)} \overbrace{H_i(A, n)^{\lambda}}^{\text{irr. comp.}} \otimes V_{\lambda}$$

and taking the Euler characteristics of these irreducible components: $\chi^{\lambda}(A, n) \in K(\mathrm{Rep}_n^{GL_n}(A))$.

APPLICATION TO NAKAJIMA QUIVER VARIETIES AND PARTITION FUNCTIONS

Nakajima quiver varieties have been intensively studied since the 90s, and they could be considered as one of the building blocks of geometric representation theory. They are defined as symplectic reductions of linear representation spaces for framed, doubled quivers: $T^*\mathrm{Rep}(Q^{fr}, \underline{\mathbf{n}})$.

The construction.

1. Dimension vectors for such quivers are split in $\underline{\mathbf{n}} = (\underline{\mathbf{v}}, \underline{\mathbf{w}})$ the dimension vectors for the original vertices and the framing vertices.
2. The moment map $\mu_{\underline{\mathbf{v}}} : T^*\mathrm{Rep}(Q^{fr}, \underline{\mathbf{n}}) \to \mathfrak{gl}_{\underline{\mathbf{v}}}$ is a general ADHM equation, and Nakajima varieties are defined as quotients of $\mu_{\underline{\mathbf{v}}}^{-1}(0)$ by $GL_{\underline{\mathbf{v}}}$.
3. The quotient can be either a quasi-projective GIT quotient $\mathcal{M}(Q, \underline{\mathbf{n}})$ (using a nontrivial character of $GL_{\underline{\mathbf{v}}}$) or affine $\mathcal{M}_0(Q, \underline{\mathbf{n}})$. They are connected by a projective morphism which is often a symplectic resolution of singularities
$$p : \mathcal{M} \to \mathcal{M}_0 \qquad (1)$$
4. For each irreducible representation $\lambda \in \mathrm{Irr}(GL_{\underline{\mathbf{v}}})$ there is a tautological sheaf $\mathcal{F}_\lambda$ on the Nakajima variety $\mathcal{M}(Q, \underline{\mathbf{n}})$, whose pushforward (in K-theory) computes the Euler characteristic of its cohomologies as
$$p_*([\mathcal{F}_\lambda]) \in K(\mathcal{M}_0) \qquad (2)$$

In the physical interpretation of Nakajima varieties as Higgs branches of 3-dimensional SUSY gauge theories, these push-forwards are used to define K-theoretic partition functions of this theory with matter fields (such as Nekrasov partition function for $Q = Q_J$, the Jordan quiver) as the formal power series:

$$Z^{(K)}(Q, \underline{\mathbf{v}}, \lambda)(q) := \sum_{\underline{\mathbf{w}}} p_*([\mathcal{F}_\lambda(\underline{\mathbf{v}}, \underline{\mathbf{w}})])q^{\underline{\mathbf{w}}}$$

Examples of Nakajima quiver varieties: For $Q = A_{n-1}$ with appropriate choice of dimension vectors

$$p : T^*\mathrm{Flag}_n \to \mathcal{N}(\mathfrak{sl}_n)$$

is the resolution of the nilpotent cone of $\mathfrak{sl}_n$. For $Q = Q_J$ the Jordan quiver, $\mathcal{M}$ is the framed moduli space of torsion free sheaves on $\mathbb{P}^2$ with rank w and second Chern class v and $\mathcal{M}_0$ is the framed moduli space of ideal instantons on $S^4 = \mathbb{C}^2 \cup \{\infty\}$.

Derived representation models.

1. As explained in Example 3. the linear space of representations of the doubled, framed quiver is $T^*\mathrm{Rep}(Q^{fr}, \underline{\mathbf{n}}) = \mathrm{Rep}_{\underline{\mathbf{n}}}(k[\overline{Q^{fr}}])$ the representation scheme of its path algebra.
2. The equation defining the moment map $\mu_{\underline{\mathbf{v}}}$ is purely algebraic, and defines an ideal in the path algebra $I_{\underline{\mathbf{v}}} \subset k[\overline{Q^{fr}}]$, so that if we consider the quotient $A := k[\overline{Q^{fr}}]/I_{\underline{\mathbf{v}}}$, we have
$$\mathrm{Rep}_{\underline{\mathbf{n}}}(A) = \mu_{\underline{\mathbf{v}}}^{-1}(0) \subset T^*\mathrm{Rep}(Q^{fr}, \underline{\mathbf{n}})$$
3. The corresponding derived partial character scheme $\mathrm{DRep}_{\underline{\mathbf{n}}}^{GL_{\underline{\mathbf{v}}}}(A)$ will be a derived affine scheme, a *derived resolution* of the affine Nakajima variety:
$$\pi_0(\mathrm{DRep}_{\underline{\mathbf{n}}}^{GL_{\underline{\mathbf{v}}}}(A)) = \mathrm{Rep}_{\underline{\mathbf{n}}}^{GL_{\underline{\mathbf{v}}}}(A) = \mathcal{M}_0$$
4. For each $\lambda \in \mathrm{Irr}(GL_{\underline{\mathbf{v}}})$, the Euler characteristic of the corresponding irreducible component of the representation homologies are now invariants in the (equivariant) K-theory of the affine Nakajima variety:
$$\chi^{\lambda}(A, \underline{\mathbf{n}}) = \chi(H_\bullet(A, \underline{\mathbf{n}})^{\lambda}) \in K_{T_{\underline{\mathbf{w}}}}(\mathcal{M}_0) \qquad (3)$$
($T_{\underline{\mathbf{w}}} \subset GL_{\underline{\mathbf{w}}}$ is the framing torus acting on these objects)

The comparison.

When forgetting the $\mathcal{O}(\mathcal{M}_0)$-module structure, Weyl's formula can be used to compute global sections of the Euler characteristics in (3) as integrals over the maximal torus $T_{\underline{\mathbf{v}}} \subset GL_{\underline{\mathbf{v}}}$. In the case of Nekrasov partition function (= Q is the Jordan quiver), for some λ these integrals have been proven ([3]) to be equal to the push-forwards in (2), the latter computed using localization formula for the framing torus $T_{\underline{\mathbf{w}}} \curvearrowright \mathcal{M}$.

It is possible to generalise this result in the case of any quiver, with some extra conditions which are satisfied in the case of the Jordan quiver.

Theorem. [S. D'Alesio - G. Felder, [4]]
If the moment map $\mu_{\underline{\mathbf{v}}}$ is flat (or equivalently its zero locus is a complete intersection of the expected dimension), then for λ in some range[a] the invariants (2) and (3) coincide

$$p_*([\mathcal{F}_\lambda]) = \chi^{\lambda}(A, \underline{\mathbf{n}}) \in K_{T_{\underline{\mathbf{w}}}}(\mathcal{M}_0) \qquad (4)$$

[a] λ must be "large enough", where the precise definition of what large means depends on the character of $GL_{\underline{\mathbf{v}}}$ used for the GIT quotient $\mathcal{M}$.

INTEGRAL FORMULAS & FURTHER RESEARCH

- When forgetting the $\mathcal{O}(\mathcal{M}_0)$-module structrure, the equality in (4) becomes an equality of numerical series. The resulting integral formulas for the Euler characteristic of tautological sheaves have the folllowing forms:
$$\mathrm{ch}_{T_{\underline{\mathbf{w}}}}(p_*[\mathcal{F}_\lambda])(t) = \int_{s \in T_{\underline{\mathbf{v}}}} \mathrm{ch}_{T_{\underline{\mathbf{v}}} \times T_{\underline{\mathbf{w}}}}(\chi(A, \underline{\mathbf{n}}))(s, t)\overline{f_\lambda(s)}\Delta(s)\, ds$$
- Further research: derived schemes for different quiver algebras, or partition functions coming from different algebras at all (Examples 1. and 2. seems to be related, respectively, to 1. the Hilbert scheme of n points in $\mathbb{A}^d$, and 2. polynomials from Macdonald-like conjectures).

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