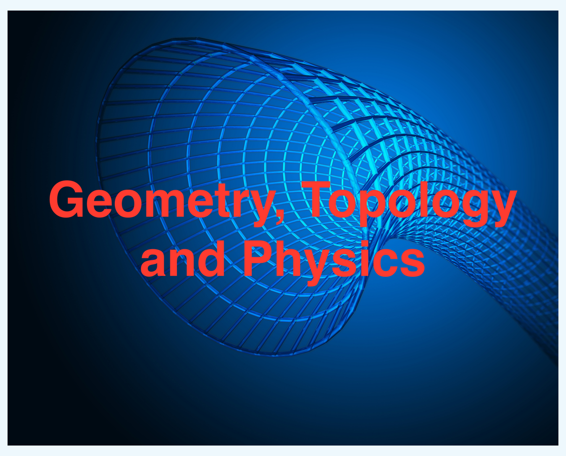

EQUIVARIANT TWISTED HKR ISOMORPHISM AND THE SMALL QUANTUM GROUP

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INTRODUCTION

This poster is about multiplicative structure of the center of the *small quantum group* (to be defined next paragraph). The center is realized in terms of certain sheaf cohomology on the Springer resolution in [1] and [2]. The present work reduces the question of the multiplicative structure of the center to the computation of certain cup-product in the sheaf cohomology of the flag varieties, which might be computable using geometric methods.

THE SMALL QUANTUM GROUP

Let $\mathfrak{g}$ be a semisimple complex Lie algebra, and $\ell \geq 3$ be an odd integer ℓ. To this data, Lusztig associated a finite-dimensional Hopf algebra $\mathfrak{u}_\ell(\mathfrak{g})$, called the *small quantum group*, which is a finite-dimensional version of a quantum group at root of unity. An important open question is a combinatorial description of the center of the small quantum group. We will focus here on the center of the principal block $\mathfrak{u}_0 \subset \mathfrak{u}_\ell(\mathfrak{g})$. Its structure is independent of ℓ.

BEZRUKAVNIKOV-LACHOWSKA'S WORK

In [2], the main result was a geometric realization of $z_0 := z(\mathfrak{u}_0)$. We state the result and will explain the notation :

Theorem 1 *There is an isomorphism of bigraded vector spaces*

$$z_0 \cong \bigoplus_{i+j+k=0} H^i(\tilde{\mathcal{N}}, \wedge^j(T\tilde{\mathcal{N}}))^k$$

Here, $\tilde{\mathcal{N}} = T^*(G/B)$ is the cotangent bundle of the flag variety G/B associated to the algebraic group of adjoint type G associated to $\mathfrak{g}$ and a Borel subgroup $B \subset G$. The subscript k comes from a grading induced by the-$\mathbb{C}^*$ action by dilation on the fibers of the projection $p : \tilde{\mathcal{N}} \to G/B$.

RESULTS

The proof of theorem 1 is based on an equivalence of derived categories

$$D^b(\mathfrak{u}_0) \cong D^b(\mathrm{Coh}^{\mathbb{C}^*}(\tilde{\mathcal{N}}))$$

and the Hochschild-Kostant-Rosenberg theorem stating that for a smooth algebraic variety X there is a vector space isomorphism

$$HH^\bullet(X) := \mathrm{Ext}^\bullet_{X\times X}(\mathcal{O}_X, \mathcal{O}_X) \cong \bigoplus_{i+j=\bullet} H^i(X, \wedge^j TX) =: HT^\bullet(X)$$

However, the natural map is usually not a ring isomorphism. The following theorem explains how to obtain a ring isomorphism :

Theorem 2 (Kontsevitch) *Let t be the Todd class of X. Then, twisting with $t^{-1/2}$ gives a ring isomorphism $HH^\bullet(X) \cong HT^\bullet(X)$.*

We verified that under a torus T action, the Todd class is T-invariant and the HKR isomorphism is T-equivariant. We obtain our main result :

Theorem 3 *[3] The composition*

$$HH^\bullet(\mathfrak{u}_0(\mathfrak{g})) \xrightarrow{HKR} \bigoplus_{i+j+k=\bullet} H^i(\tilde{\mathcal{N}}, \wedge^j T\tilde{\mathcal{N}})^k \xrightarrow{\langle -, Todd(\tilde{\mathcal{N}})^{-1/2}\rangle} \bigoplus_{i+j+k=\bullet} H^i(\tilde{\mathcal{N}}, \wedge^j T\tilde{\mathcal{N}})^k$$

is a ring isomorphism.

Corollary 1 *Twisting by the Todd class of $\tilde{\mathcal{N}}$ gives a ring isomorphism*

$$z_0 \cong \bigoplus_{i+j+k=0} H^i(\tilde{\mathcal{N}}, \wedge^j(T\tilde{\mathcal{N}}))^k$$

The theorem 3 essentially follows from these two propositions (here X is a smooth complex algebraic variety acted upon by a torus T) :

Proposition 1 *[3] In the derived category $D^b(\mathrm{Coh}(X))$, the quasi-isomorphism*

$$\iota^*\mathcal{O}_\Delta \cong \bigoplus_{i\in X} \Omega^i_X[i]$$

is T-equivariant.

Here $\iota : \Delta \to X \times X$ is the inclusion map.

Proposition 2 *[3] Let $t \in H\Omega^\bullet(X)$ be the Todd class of X. Then t is T-invariant.*

In particular twisting the HKR isomorphism with $t^{-1/2}$ gives a T-equivariant multiplicative isomorphism $HH^\bullet(X) \cong HT^\bullet(X)$. We hope that a similar statement holds where T is replaced by a reductive group G.

EXAMPLE

Let $\mathfrak{g} = \mathfrak{sl}_3$. The bigraded dimensions of z_0 are as follows :

$j-i$	0	2	4	6
$i+j=0$	1			
$i+j=2$	2	1		
$i+j=4$	2	3	1	
$i+j=6$	1	2	2	1

Geometrically, the first colum is canonically isomorphic to the cohomology of the flag variety, and the big diagonal correspond to the subalgebra spanned by the Poisson bivector field $\tau \in H^0(\tilde{\mathcal{N}}, \wedge^2 T\tilde{\mathcal{N}})$. An easy geometric argument shows :

Proposition 3 *[3] The subalgebra generated by $H^*(G/B)$ and τ is untwisted.*

For $\mathfrak{sl}_3$, it describes the multiplicative structure of a codimension 1 subalgebra of z_0.

REFERENCES

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