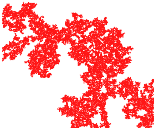
Russo-Seymour-Welsh theory for planar percolation



Vincent Tassion

ETH zürich · erc European Research Council · SwissMAP The Mathematics of Physics National Centre of Competence in Research

SwissMAP meeting, September 9, 2020

Percolation: how does a fluid propagate in a random medium?

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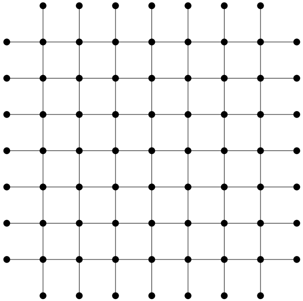
How does water flow in rocks?

How do fires propagate in forests?

Bernoulli percolation [Broadbent and Hammersley, 1957]

$(\mathbb{Z}^2, E(\mathbb{Z}^2))$



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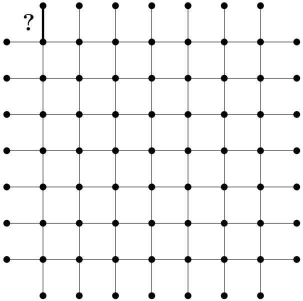

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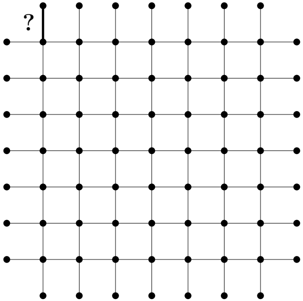

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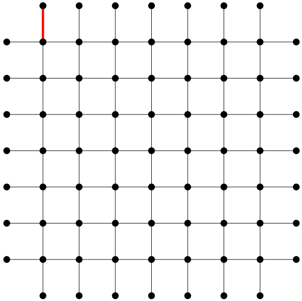

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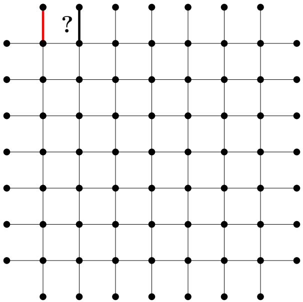

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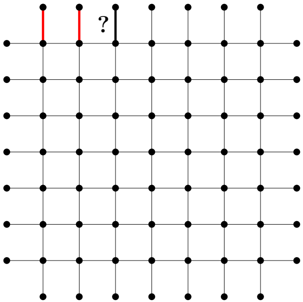

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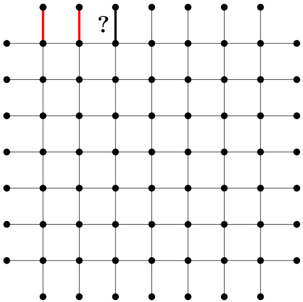

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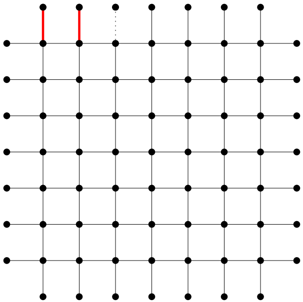

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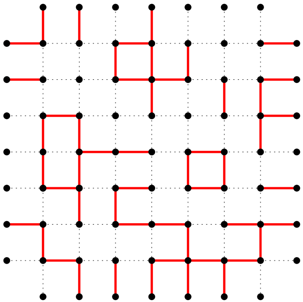

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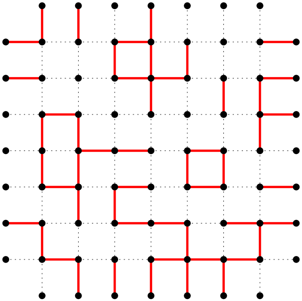

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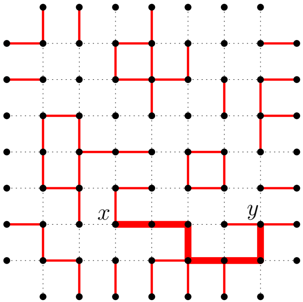

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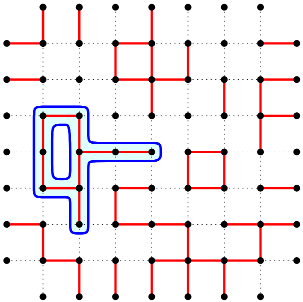

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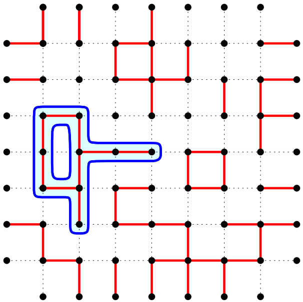

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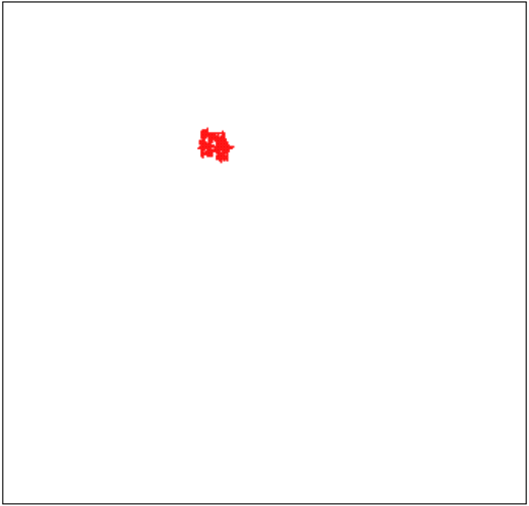
QUESTION: Is there an infinite cluster?

Largest cluster in a box for Bernoulli percolation on $\mathbb{Z}^2$



$p < p_c$

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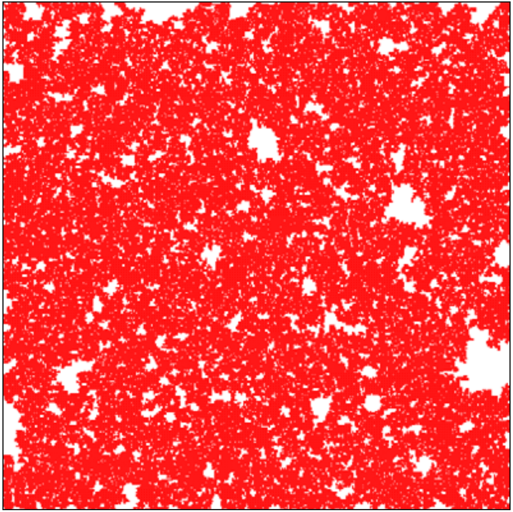

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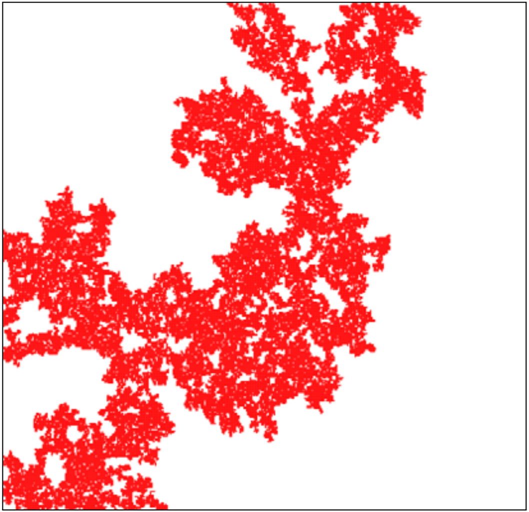
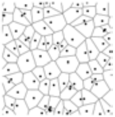


$p = p_c$

Interactions with other fields

Interactions with other fields

Stochastic geometry



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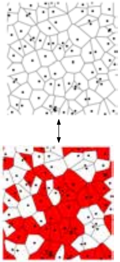

Voronoi percolation

[Vahidi-Asl Wierman '90]

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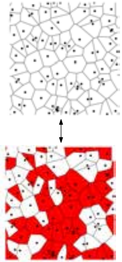

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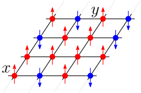
Spin systems



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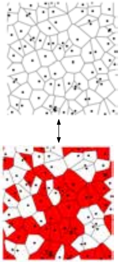

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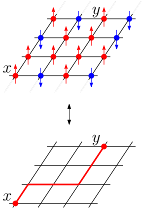

FK percolation

[Fortuin Kasteleyn 74]

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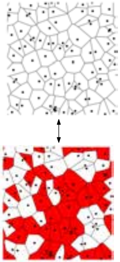

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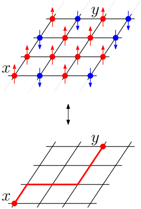

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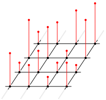

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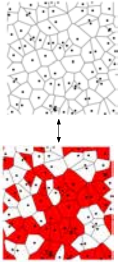

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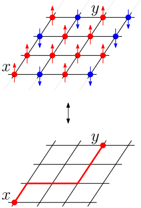

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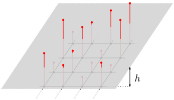

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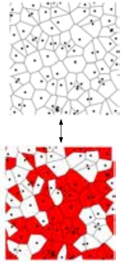

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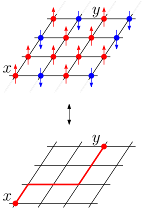

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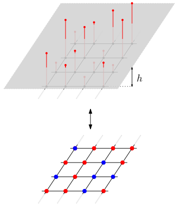

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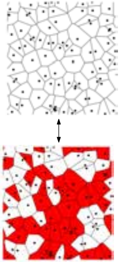

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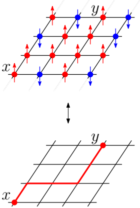

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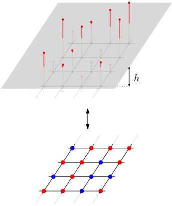

GFF percolation
Nodal lines

Outlook of the talk

1. RSW theory for Bernoulli percolation on $\mathbb{Z}^2$

[Russo '78][Seymour Welsh '78]

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1. RSW theory for Bernoulli percolation on $\mathbb{Z}^2$

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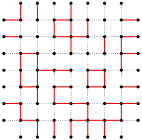
↓

2. RSW theory for dependent planar models

[Köhler-Schindler '20+]

1. RSW theory for Bernoulli percolation on $\mathbb{Z}^2$.



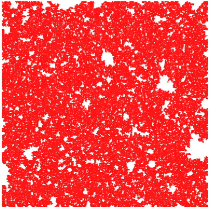
Phase transition in dimension 2

Theorem [Kesten '80]

For Bernoulli percolation on $\mathbb{Z}^2$, we have

$$p_c = \frac{1}{2}.$$

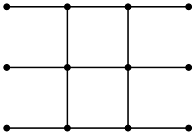
$p < \frac{1}{2}$



$p > \frac{1}{2}$

What is special about $p = 1/2$?

Planar duality:

G 

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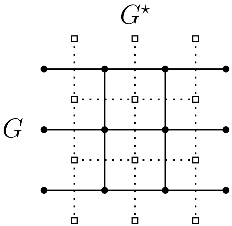

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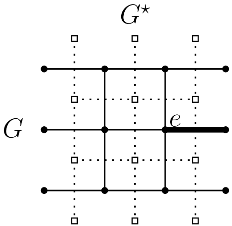

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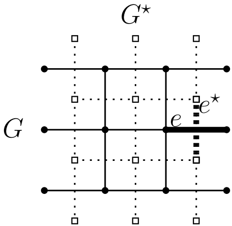

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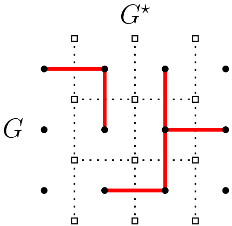

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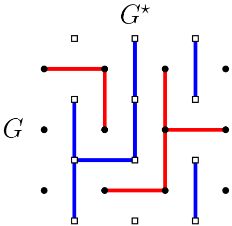

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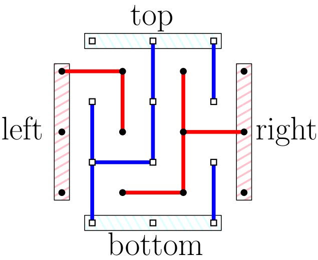

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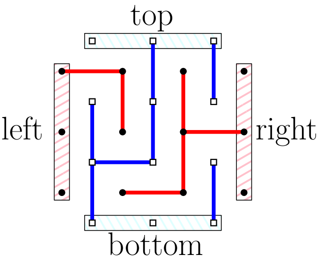
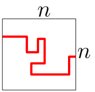
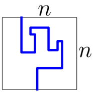


Consequence: For every n, $\mathrm{P}_p\Big[$ 

$\Big] + \mathrm{P}_p\Big[$ 

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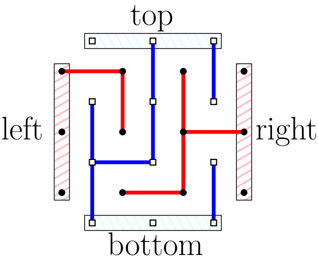
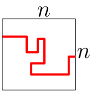
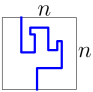


Consequence: For every n, $\mathrm{P}_p\Big[$ 

$\Big] + \mathrm{P}_p\Big[$ 

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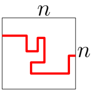
➜ For $p = 1/2$, $\mathrm{P}_p\Big[$ 

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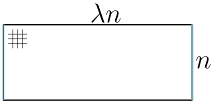
Crossing probabilities

- **A rectangle of aspect ratio** $\lambda > 0$:

$R =$ 

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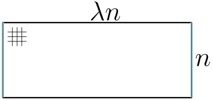
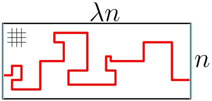
$R =$ 

- **A crossing probability at parameter** p:

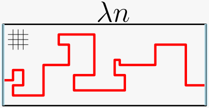
$\mathrm{P}_p\Big[$ 

$\Big] = \mathrm{P}_p\Big[$ There exists an open path in R from left to right. $\Big]$.

Cardy's formula

Conjecture: critical behavior of the crossing probabilities

Fix $\lambda > 0$. For critical Bernoulli percolation on $\mathbb{Z}^2$, we have

$$\lim_{n\to\infty} \mathrm{P}_{p_c}\Big[$$


$$n\Big] = \underbrace{f(\lambda)}_{\text{Cardy's formula}}.$$

Properties of Cardy's formula:

Cardy's formula

Conjecture: critical behavior of the crossing probabilities

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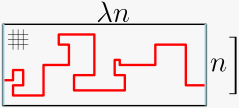

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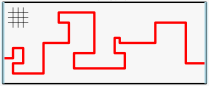

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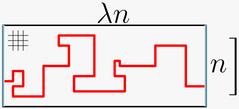

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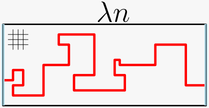

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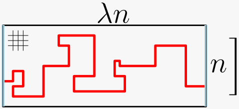

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For the triangular lattice:

- Cardy's formula and conformal invariance, [Smirnov '01]

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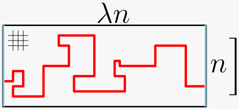

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- Critical exponents

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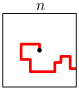

$$\Bigg] = n^{-5/48+o(1)}.$$ [Lawler Schramm Werner '02]

Russo-Seymour-Welsh theory

Consider Bernoulli percolation on $\mathbb{Z}^2$ at $p_c = 1/2$.

RSW theorem [Russo 78] [Seymour Welsh 78]

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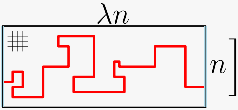
$c(\lambda) \leqslant \mathrm{P}_{p_c}\Big[$ 

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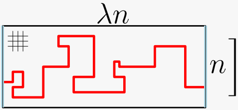

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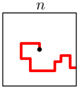

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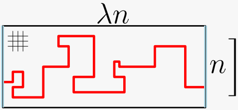

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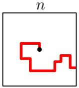

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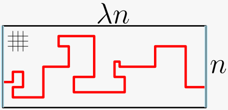
$$c(\lambda) \leqslant \mathrm{P}_{p_c}\Big[$$ 

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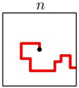
  ➜ $n^{-c_1} \leqslant \mathrm{P}_{p_c}\Big[$ 

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- Study of near-critical regime ($p = p_c \pm \varepsilon$),
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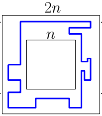
Annulus argument

RSW theorem: $\mathrm{P}_{1/2}\Big[$ 

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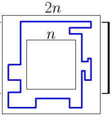
RSW theorem: $\mathrm{P}_{1/2}\Bigg[$ 

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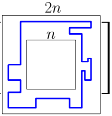
RSW theorem: $\mathrm{P}_{1/2}\Bigg[$  $\Bigg] \geqslant c.$

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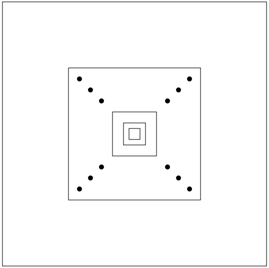
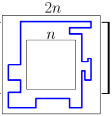


Annulus argument

RSW theorem: $\mathrm{P}_{1/2}$ [] $\geqslant c$.

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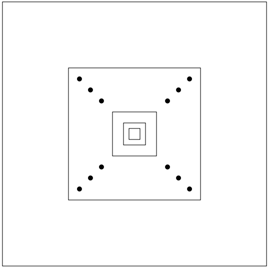
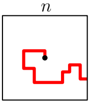
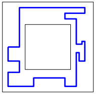


P_p [] $\leqslant \mathrm{P}_p$ [NO ]

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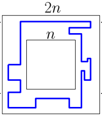
RSW theorem: $\mathrm{P}_{1/2}\Big[$  $\Big] \geqslant c.$

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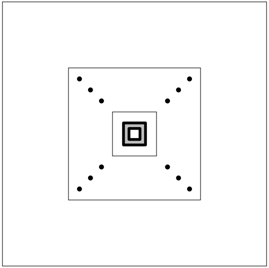
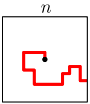
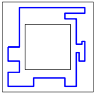


$\mathrm{P}_p\Big[$  $\Big] \leqslant \mathrm{P}_p\Big[\mathrm{NO}$  $\Big]$

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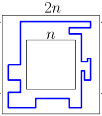
RSW theorem: $\mathrm{P}_{1/2}\Big[$  $\Big] \geqslant c.$

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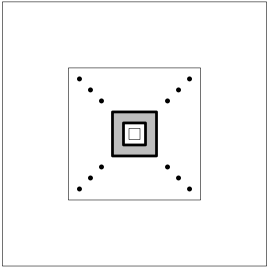
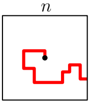
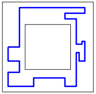


$\mathrm{P}_p\Big[$  $\Big] \leqslant \mathrm{P}_p\Big[\mathrm{NO}$  $\Big]$

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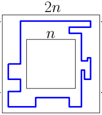
RSW theorem: $\mathrm{P}_{1/2}\Big[$  $\Big] \geqslant c.$

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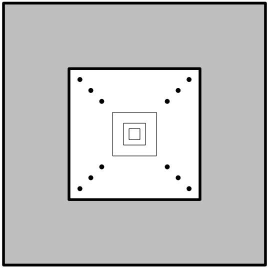
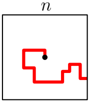
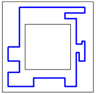


$\mathrm{P}_p\Big[$  $\Big] \leqslant \mathrm{P}_p\Big[\mathrm{NO}$  $\Big]$

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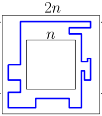
RSW theorem: $\mathrm{P}_{1/2}\Big[$  $\Big] \geqslant c.$

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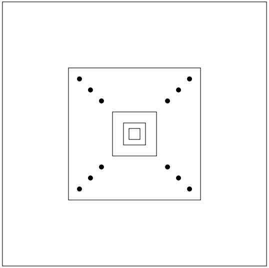
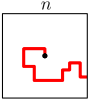
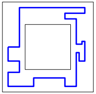


$\mathrm{P}_p\Big[$  $\Big] \leqslant \mathrm{P}_p\Big[\mathrm{NO}$  $\Big]$

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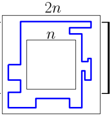
RSW theorem: $\mathrm{P}_{1/2}\Big[$ 

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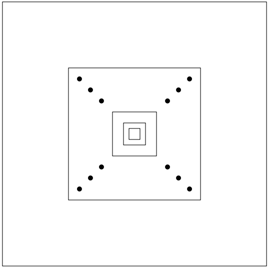
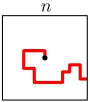
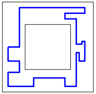


$\mathrm{P}_p\Big[$ 

$\Big] \leqslant \mathrm{P}_p\Big[\mathrm{NO}$  $\Big] \leqslant (1-c)^{\log_2 n}$

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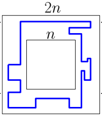
RSW theorem: $\mathrm{P}_{1/2}\Bigg[$  $\Bigg] \geqslant c.$

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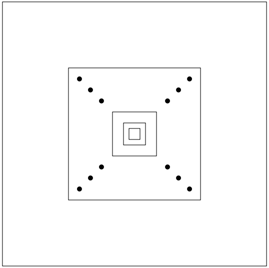
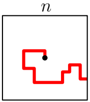
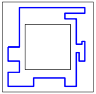
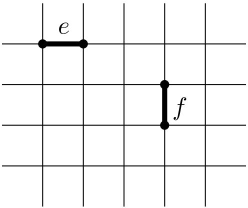
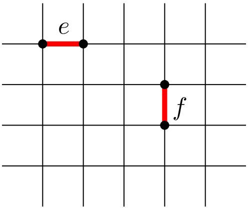


$\mathrm{P}_p\Bigg[$  $\Bigg] \leqslant \mathrm{P}_p\Bigg[\mathrm{NO}$  $\Bigg] \leqslant (1-c)^{\log_2 n} \leqslant \dfrac{1}{n^{c'}}.$

Bernoulli percolation: an independent percolation model

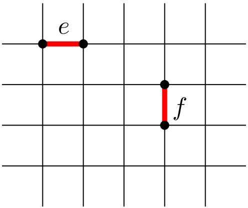


Bernoulli percolation: an independent percolation model



$\mathrm{P}_p[e, f \text{ open}] = p^2$

Bernoulli percolation: an independent percolation model



$$\boxed{\mathrm{P}_p[e, f \text{ open}] = p^2 = \mathrm{P}_p[e \text{ open}]\mathrm{P}_p[f \text{ open}].}$$

1. RSW theory for Bernoulli percolation on $\mathbb{Z}^2$

[Russo '78][Seymour Welsh '78]

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2. The RSW lemma and its generalizations

[Köhler-Schindler T. 20+]

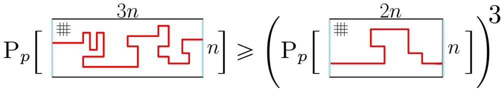
Two important properties of P_p

Symmetries:

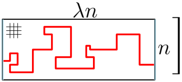
P_p is invariant under translations, reflections and $\pi/2$-rotation.

Positive correlations [Harris 60]:

Crossing events are positively correlated.

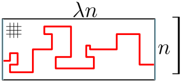


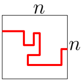
Proof of the RSW theorem

Goal: For $\lambda \geqslant 1$ and $n \geqslant 1$, $\mathrm{P}\Big[$ 

$\Big] \geqslant c(\lambda).$

Proof of the RSW theorem

Goal: For $\lambda \geqslant 1$ and $n \geqslant 1$, $\mathrm{P}\Big[$  $\Big] \geqslant c(\lambda)$.

Step 0 (self-duality): $\mathrm{P}\Big[$  $\Big] = 1/2$.

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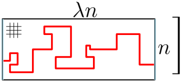
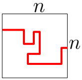
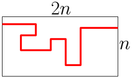
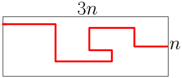
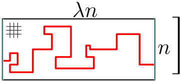
Goal: For $\lambda \geqslant 1$ and $n \geqslant 1$, $\mathrm{P}\Big[$  $\Big] \geqslant c(\lambda)$.

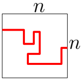
Step 0 (self-duality): $\mathrm{P}\Big[$  $\Big] = 1/2$.

Step 2 ($\lambda = 2$ suffices): $\mathrm{P}\Big[$  $\Big] \geqslant c \overset{\text{FKG}}{\Rightarrow} \mathrm{P}\Big[$  $\Big] \geqslant c^3$.

Proof of the RSW theorem

Goal: For $\lambda \geqslant 1$ and $n \geqslant 1$, $\mathrm{P}\Big[$ 

$\Big] \geqslant c(\lambda)$.

Step 0 (self-duality): $\mathrm{P}\Big[$ 

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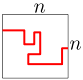
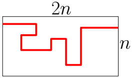
Step 1 (**RSW Lemma**): $\mathrm{P}\Big[$ 

$\Big] \geqslant c \Rightarrow \mathrm{P}\Big[$ 

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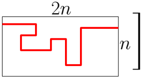
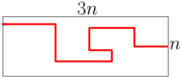
Step 2 ($\lambda = 2$ suffices): $\mathrm{P}\Big[$ 

$\Big] \geqslant c \overset{\text{FKG}}{\Rightarrow} \mathrm{P}\Big[$ 

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The proof of RSW theorem: The key lemma

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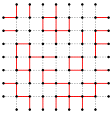
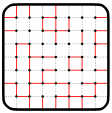
$\mathbf{P}$=general percolation process in the plane.



Bernoulli percolation

[Broadbent Hammersley, 57]



FK percolation

[Fortuin Kasteleyn 74]

Voronoi percolation

[Vahidi-Asl Wierman '90]

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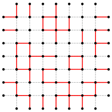

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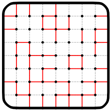

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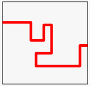
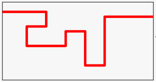
RSW Lemma

$\Big(\mathbf{P}\Big[$ 

$n\Big] \geqslant c\Big) \Rightarrow \Big(\mathbf{P}\Big[$ 

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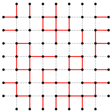

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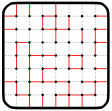

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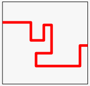
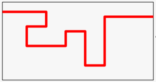
RSW Lemma

n

$\Big(\mathbf{P}\Big[$  $n\Big] \geqslant c\Big) \Rightarrow$

$2n$

$\Big(\mathbf{P}\Big[$  $n\Big] \geqslant c'\Big),$

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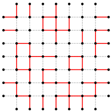

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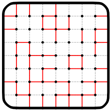

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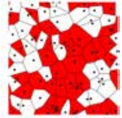

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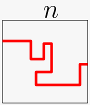
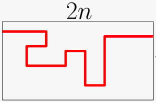
RSW Lemma

$\Big(\mathbf{P}\Big[$ 

$n\Big] \geqslant c\Big) \Rightarrow \Big(\mathbf{P}\Big[$ 

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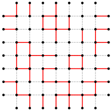

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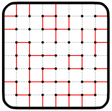

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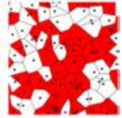

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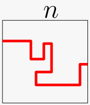
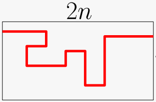
$\Big(\mathbf{P}\Big[$ 

$n\Big] \geqslant c\Big) \Rightarrow \Big(\mathbf{P}\Big[$ 

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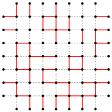

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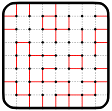

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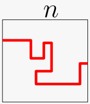
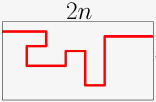
$\Big(\mathbf{P}\Big[$  $n\Big] \geqslant c\Big) \Rightarrow \Big(\mathbf{P}\Big[$  $n\Big] \geqslant c'\Big),$

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All the proofs use **Symmetries** + **Positive correlations** + "something else".

General RSW Lemma.

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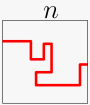
RSW Lemma for symmetric positively correlated measures

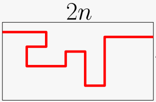
[Köhler-Schindler T. 20+]

Let $\mathbf{P}$ be a planar percolation measure satisfying

- Symmetries,
- Positive correlations.

Then

$\Bigg(\mathbf{P}\Bigg[$ 

$n\Bigg] \geqslant c\Bigg) \Rightarrow \Bigg(\mathbf{P}\Bigg[$ 

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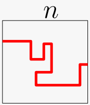
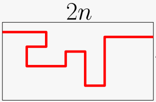
$\Big(\mathbf{P}\Big[$ 

$n\Big] \geqslant c\Big) \Rightarrow \Big(\mathbf{P}\Big[$ 

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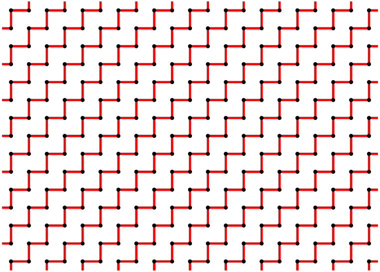
Why are symmetries important?



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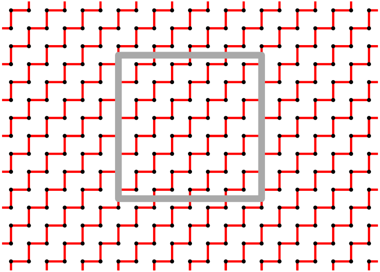

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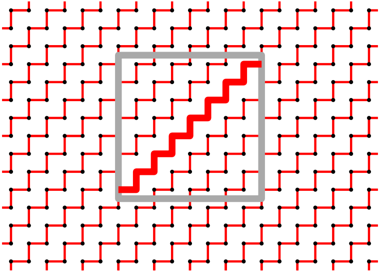

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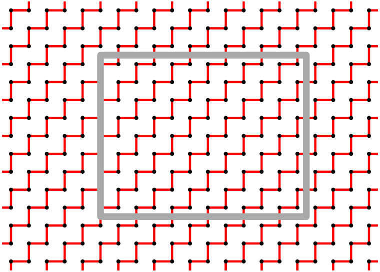

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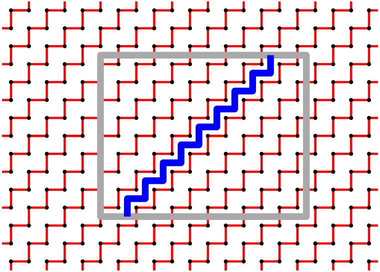

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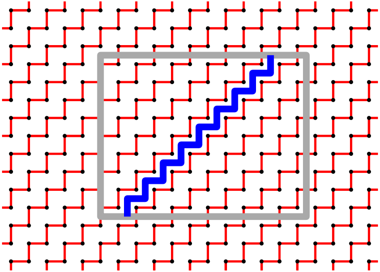

➜ Squares always crossed, long horizontal rectangles never crossed,

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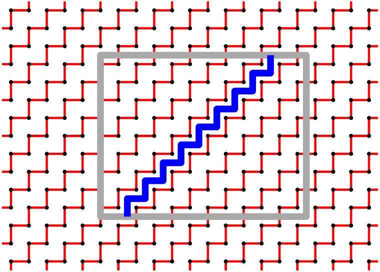
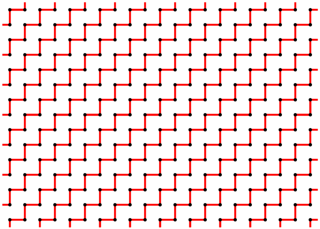
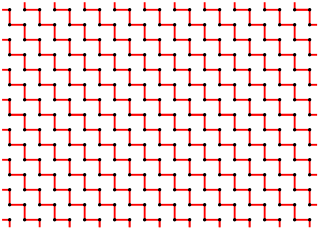


➜ Squares always crossed, long horizontal rectangles never crossed,

➜ Not reflection invariant.

Why are positive correlations important?

$\frac{1}{2}$  $+ \frac{1}{2}$ 

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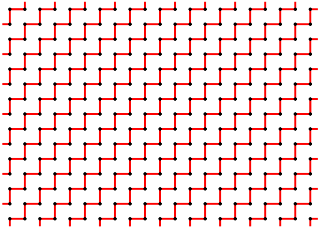
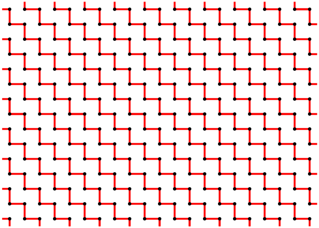
$\frac{1}{2}$  $+\ \frac{1}{2}$ 

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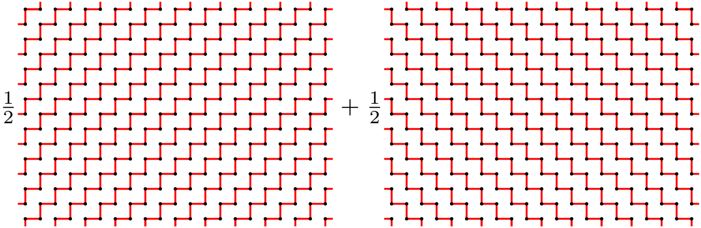


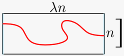
➜ Squares always crossed, long horizontal rectangles never crossed,

➜ No positive correlation.

Conclusion and outlook

Critical behavior of Bernoulli percolation

✔ Robust RSW theory:

$c(\lambda) \leqslant \mathrm{P}_{p_c}\big[$ 

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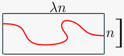
$$c(\lambda) \leqslant \mathrm{P}_{p_c}\Big[$$ 

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Perspectives:

Conclusion and outlook

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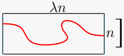
$$c(\lambda) \leqslant \mathrm{P}_{p_c}\Big[$$ 

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Conclusion and outlook

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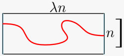
$$c(\lambda) \leqslant \mathrm{P}_{p_c}\Big[$$ 

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- RSW for non positively correlated models.
- Prove Cardy's formula for some specific models.

Conclusion and outlook

Critical behavior of Bernoulli percolation

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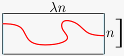
$c(\lambda) \leqslant \mathrm{P}_{p_c}\big[$ 

$\big] \leqslant 1 - c(\lambda).$

Perspectives:

- RSW for non positively correlated models.
- Prove Cardy's formula for some specific models.
- Universality and scaling limit.

Conclusion and outlook

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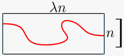
$c(\lambda) \leqslant \mathrm{P}_{p_c}\big[$ 

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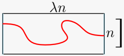
$c(\lambda) \leqslant \mathrm{P}_{p_c}\big[$  $\big] \leqslant 1 - c(\lambda).$

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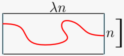
$c(\lambda) \leqslant \mathrm{P}_{p_c}\Big[$ 

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