Einstein's Field Equations

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Basic Notions
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Lorentzian geometry 1

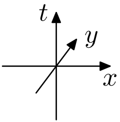
Spacetime:

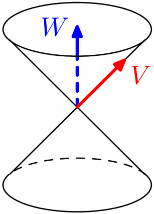
- M: 4-dimensional manifold
- g: Lorentzian metric $(-+++)$ on M

Example: Minkowski space. (Setting of Special Relativity.)

$$M = \mathbb{R}^4 \ni (t, x, y, z), \qquad g = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

($\mathbb{R}^4$ with quadratic form $\mathrm{diag}(-1, 1, 1, 1)$.)





$W = 2\partial_t$ is timelike: $g(W, W) < 0$.

$V = \partial_t + \partial_x$ is lightlike: $g(V, V) = 0$.

Lorentzian metric: $g(z) = \sum_{i,j=1}^4 g_{ij}(z)\mathrm{d}z^i\,\mathrm{d}z^j$, signature of $(g_{ij}(z))$ is $(3, 1)$.

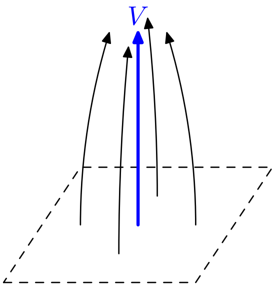
Lorentzian geometry 2

Curvature at $p \in M$: Riemann curvature tensor $R(V, W, X, Y)$, Ricci tensor $\mathrm{Ric}(V, W)$, scalar curvature S

Einstein tensor: $\mathrm{Ein}(g) = \mathrm{Ric}(g) - \frac{1}{2}S(g)g$

Roughly: For a timelike vector V, $\mathrm{Ric}(V, V)$ is a measure of concentration of freely falling observers near V. (Cf. tidal forces.)

(Freely falling observers: curves in (M, g) with zero acceleration.)



Einstein's Field Equations

- M: 4-dimensional manifold
- g: Lorentzian metric $(-+++)$ on M

For vacuum spacetimes:

$$\boxed{\mathrm{Ein}(g) = 0} \quad \text{or equivalently} \quad \boxed{\mathrm{Ric}(g) = 0}.$$

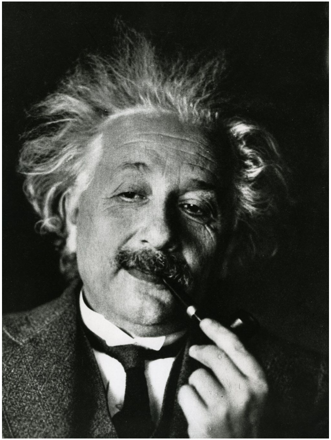
In local coordinates, $\mathrm{Ric}(g)_{ij} = -\frac{1}{2}(g^{-1})^{kl}\partial_k\partial_l g_{ij} + \dots$.
Quasilinear second order PDE, 'sort of' a wave equation.

Interaction of gravity and matter/energy:

$$\boxed{\mathrm{Ein}(g) = T} \qquad \text{— Einstein (1915).}$$

where $T = T(V, W)$ is the stress-energy-momentum tensor.
($T(V, V)$ is the energy density of the energy/matter as observed by an observer traveling along the unit timelike vector V.)

$$
\begin{aligned}
R_{ij} = & -\frac{1}{2}\sum_{a,b=1}^{n}\left(\frac{\partial^2 g_{ij}}{\partial x^a \partial x^b} + \frac{\partial^2 g_{ab}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{ib}}{\partial x^j \partial x^a} - \frac{\partial^2 g_{jb}}{\partial x^i \partial x^a}\right) g^{ab} \\
& + \frac{1}{2}\sum_{a,b,c,d=1}^{n}\left(\frac{1}{2}\frac{\partial g_{ac}}{\partial x^i}\frac{\partial g_{bd}}{\partial x^j} + \frac{\partial g_{ic}}{\partial x^a}\frac{\partial g_{jd}}{\partial x^b} - \frac{\partial g_{ic}}{\partial x^a}\frac{\partial g_{jb}}{\partial x^d}\right) g^{ab} g^{cd} \\
& - \frac{1}{4}\sum_{a,b,c,d=1}^{n}\left(\frac{\partial g_{jc}}{\partial x^i} + \frac{\partial g_{ic}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^c}\right)\left(2\frac{\partial g_{bd}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^d}\right) g^{ab} g^{cd}.
\end{aligned}
$$

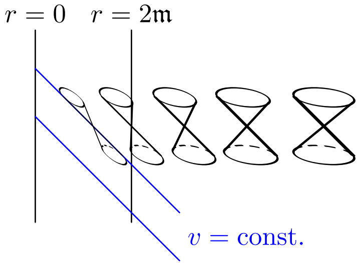


Examples

Vacuum spacetimes: $\mathrm{Ein}(g) = 0$ ($\Leftrightarrow \mathrm{Ric}(g) = 0$)

1. Minkowski space (1908):
   $(M, g) = (\mathbb{R}^4, -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$
2. Schwarzschild black hole (1916): parameter $\mathfrak{m} > 0$ (mass)

$$g = -(1 - 2\mathfrak{m}/r)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r^2 + r^2 g_{\mathbb{S}^2}$$



3. Gravitational waves propagating on a given spacetime (M, g):
   solutions of '$\mathrm{Ein}(g + \delta g) = 0$', i.e. $D_g \mathrm{Ein}(\delta g) = 0$

Further examples

Cosmological constant $\Lambda \in \mathbb{R}$:

$$\boxed{\mathrm{Ein}(g) = -\Lambda g + T.}$$

Vacuum spacetimes ($T = 0$) with $\Lambda \neq 0$:

1. Anti–de Sitter space ($\Lambda < 0$):

$$g = (3/|\Lambda|)x^{-2}\big(-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2\big), \qquad t \in \mathbb{R},\ x > 0,\ y \in \mathbb{R}^2$$

2. de Sitter space ($\Lambda > 0$)

$$g = (3/\Lambda)\tau^{-2}\big(-\mathrm{d}\tau^2 + \mathrm{d}x^2\big), \qquad \tau > 0,\ x \in \mathbb{R}^3.$$

3. ...

Hyperbolic PDE (vacuum case)

- If $\mathrm{Ric}(g) = 0$, then also $\mathrm{Ric}(\phi^* g) = 0$ for any diffeomorphism (change of coordinates) $\phi \colon M \to M$.
- Fix any coordinate system (x^0, x^1, x^2, x^3) on M. Demand 4 extra conditions on g, e.g. $\Box_g x^j = 0$. (Can always be arranged by replacing g by $\phi^* g$ for carefully chosen ϕ.)
- Solving $\mathrm{Ric}(g) = 0$ together with $\Box_g x^j = 0$ can be phrased as a single quasilinear wave equation, schematically

$$\Box_g g_{ij} = N_{ij}(g, \partial g)$$

- Initial value problem! Choquet-Bruhat (1952), Geroch (1969). Starting point (analysis & numerics) for studying
  - dynamics of spacetimes (perturbation theory, strong field regimes, ...)
  - construction of 'interesting' solutions from 'interesting' initial data (e.g. black hole collisions/mergers)
  - ...