Introduction to cosmology and general relativity
Exercises and activities

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Contents

1 Parameters

Exercise 1: The light year

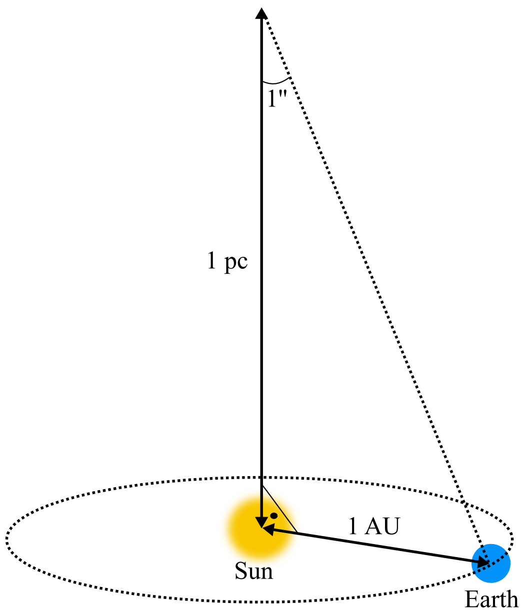
(a) Define what a ***light year*** (ly) is.

(b) Knowing that light propagates in a vacuum at a velocity of $c = 299'792'458$ m/s, express 1 ly in meters. Express this result with a single meaningful digit that is easy to remember.

(c) "Looking far away means looking into the past": why? When observing galaxies further and further away, the proportion of irregular galaxies increases compared to spiral galaxies. How can this observation be explained? Formulate some hypotheses.

Exercise 2: The parsec

The **parsec**, abbreviated **pc**, is the most widely used unit of distance measurement in astronomy. It is based on the trigonometric parallax, the oldest and most reliable method of measuring stellar distances. Consider the right-angled triangle in the figure below, which has as its first side of the right angle the ***astronomical unit (AU)***, which is the radius of the Earth's orbit around the Sun, opposite to the angle of $1.0''$ ($= 1°/3600$). The parsec is defined as the other side of the right angle of this triangle.

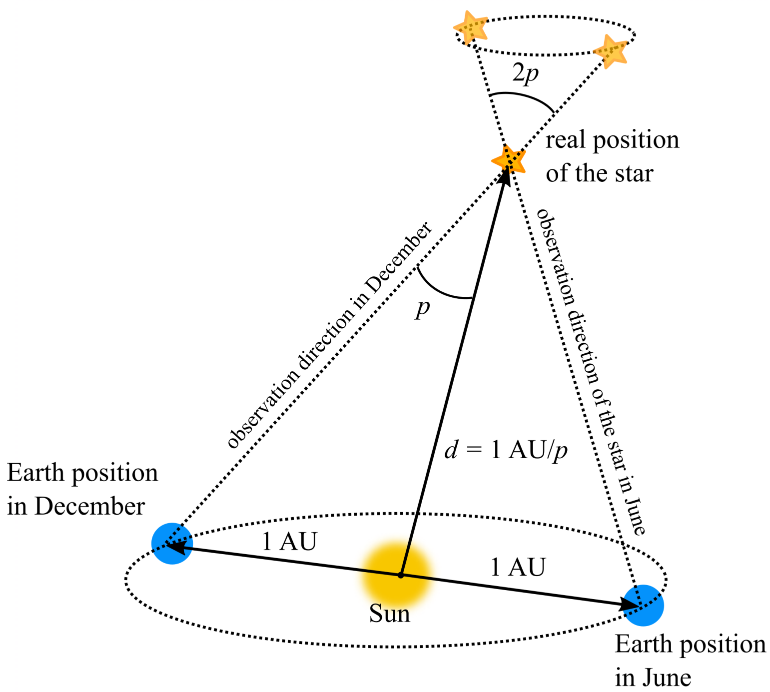


a) Knowing that 1 AU $= 1.496 \cdot 10^{11}$ m, express 1 pc in m.

The drawing below shows how the apparent position of a star in the sky changes due to the Earth's rotation around the Sun. The angle p is called the ***parallax*** of the star.

b) Determine the formula expressing the distance d between a star and the Sun (in pc) as a function of its parallax in arc seconds: $d(p)$.

Be careful: the farther away a star is, the smaller the parallax angle p. Since there is no star closer than one parsec, p is always smaller than $1°$, and we can assume
1) that the Earth-star distance matches the Sun-star distance, d,
2) that $\tan(p) \approx p$.



c) The nearest star, Proxima Centauri, is 4.23 ly from the Solar System. What is its distance in pc? And its parallax?

d) If the uncertainty on the parallax measurement of a source is in the order of $0.001''$, what is the order of magnitude of the maximum distance we can estimate by this method? What does this order of magnitude correspond to?

e) Do you know of methods for determining distances to more distant objects?

Video distance OM: http://apod.nasa.gov/apod/ap150324.html

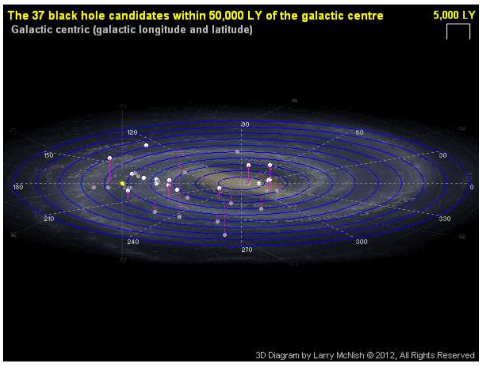
Exercise 3: How many stars?

From the size of the Milky Way and the average distance between stars in its disk, estimate the number of stars in it, assuming that the bright part of the Milky Way is a cylinder of thickness $h = 0.3$ kpc.

Proxima Centauri is a small red star, visible only with a telescope. It is part of the "Alpha Centauri" star system, visible from the southern hemisphere, whose brightest star is similar to the Sun. Credit: http://apod.nasa.gov/apod/ap160118.html

Exercise 4: Average distance between black holes in the Milky Way

It is estimated that the Milky Way contains about a hundred million stellar black holes, even in its halo[1]. From this information, estimate the order of magnitude (OM) of the average distance between stellar black holes in the Milky Way.



Distribution of sources that could be black holes within 50'000 ly from the galactic center. Credit: https ://calgary.rasc.ca/blackholes.htm.

[1] Credit: A. Barrau, *Trous noirs et espace-temps*, Bayard (2022).

Exercise 5: Average light densities

a) Using a handbook of physics formulas (or searching the Internet), calculate the average density of the Sun, Earth, and then the Solar System (in scientific notation and in SI units, with a relevant number of significant digits).

b) Studies of nearby stars show that the set of visible stars has, on average, a mass to brightness ratio five times that of the Sun[2]

$$\frac{M_{\text{lum}}}{L_{\text{lum}}} = 5\,\frac{M_\odot}{L_\odot}$$

where $\mathrm{M}_\odot$ and $\mathrm{L}_\odot$ represent, respectively, the mass and luminosity of the Sun. The luminosity of the Galaxy is $L_{\text{Gal}} \approx 2 \cdot 10^{11}\ \mathrm{L}_\odot$, and its radius, taking into account only the luminous matter, is about 15 kpc. Estimate the mass of luminous matter of the Milky Way, $M_{\text{lum Gal}}$, and then its average luminous density, $\rho_{\text{lum Gal}}$, (results in scientific notation and in SI units, with a relevant number of significant digits).

c) We count an average of 5 galaxies per cubic portion of the universe with 10 Mpc sides. Assuming that the luminous mass of the Milky Way is representative of the average luminous masses of all galaxies, deduce the average luminous density of the universe. Express this result in $\mathrm{M}_\odot/\mathrm{Mpc}^3$ and in $\mathrm{kg/m}^3$, in scientific notation. What is its order of magnitude? Does this result agree with the value given in the course?

d) Convert this density to the number of hydrogen atoms per cubic meter.

[2] http://roffet.com/documents/sciences/mise-en-evidence-de-la-masse-cachee/i/

Exercise 6: Collision probability

The image below was created from NASA simulation data indicating that the Milky Way and Andromeda are approaching and could collide in about four billion years. This is a reconstruction of what the sky might look like to us from Earth at that time.

Credit: http://www.nasa.gov/mission_pages/hubble/science/milky-way-collide.html (NASA).

Video of the simulation of the collision between the Milky Way and Andromeda:
https://www.youtube.com/watch?v=kvyeP_bI4bc

We can consider the bright disk of Andromeda as a circular surface of radius $R \approx 30$ kpc, containing about 400 billion stars. We assume that the average radius of the stars is equal to that of the Sun: $r \approx 10 \cdot \mathrm{R}_{\odot}$.

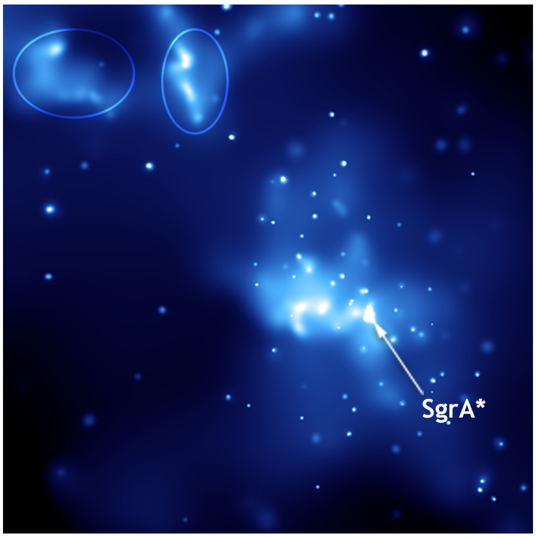
a) Explain why collisions between galaxies are quite frequent, while collisions between stars of two colliding galaxies are rare.

b) Estimate the surface percentage occupied by Andromeda stars compared to the total area occupied by the galactic disk. This percentage corresponds to the probability of a Milky Way star (e.g., the Sun) colliding with an Andromeda star during the collision between these two galaxies.

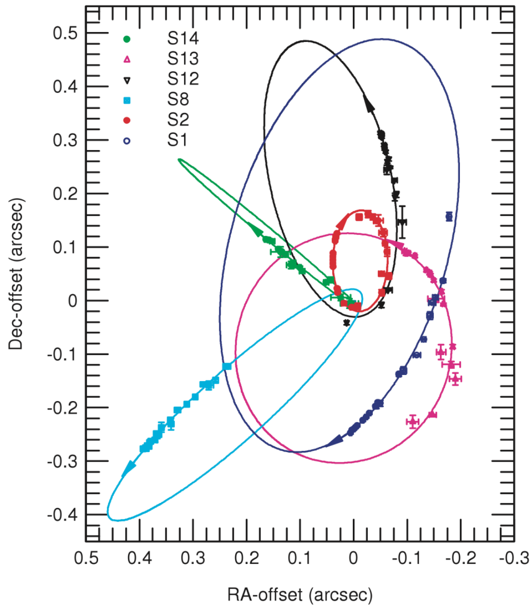
c) What is the order of magnitude of the total number of stellar collisions in the collision between Andromeda and the Milky Way?

Exercise 7: The orbit of the S2 star

The figure below shows an image of Sagittarius A* (Sgr A*), the compact radio source at the center of the Milky Way. It was taken with the NASA X-ray telescope. The images in the ellipses are echoes, reflections of X-ray radiation on nearby clouds. By studying the motion of some stars around this source, astronomers were able to infer the presence of a supermassive black hole at this spot, as well as in all large spiral galaxies.



Credit: NASA, Wikipedia



Credit: Eisenhauer et al., 2005, ApJ 628, 246

The image on the right shows the orbits of six stars around Sgr A*. The star S2, whose entire orbit we were able to follow from 1995 to 2010, has a perihelion (the minimum distance from Sgr A* in its orbit) of 120 AU and a period of 15 years. To simplify, we consider S2's orbit as a circle of radius r equal to 1000 AU (about ten times larger than its perihelion).

a) Using this approximation, estimate the average scalar velocity of S2 during a period. Give the result in SI units.

b) Using a handbook of physics formulas (or searching the Internet), calculate the distance Neptune travels during a complete orbit around the Sun (use the circular orbit approximation).

c) Compare the estimated lengths of the S2 and Neptune orbits by calculating their ratio.

d) Calculate the average scalar velocity of Neptune on a full orbit around the Sun.

e) Compare the average scalar velocities of S2 and Neptune by calculating their ratio.

A simulation of S2's orbit can be viewed at this address:
https://phys.org/news/2017-08-stars-orbiting-supermassive-black-hole.html#jCp

Exercise 8: Comparison of gravitational and electrical interactions

First part:

According to Bohr's model, the electron in the hydrogen atom revolves around the proton[3] along an orbit of radius $r = 5,3 \cdot 10^{-11}$ m.

a) Calculate the intensity of the electron – proton gravitational force of attraction. Then, in the same atom, calculate the intensity of the electric force of attraction between the electron and the proton. The charge and mass values of these particles, as well as the constants G (universal gravitational constant) and k (Coulomb constant) are given in Appendix A of this course.

b) What is the relationship between the intensity of the electric force and the gravitational force in the hydrogen atom? Does this relationship change if we vary the distance between the two particles?

Part two:

c) Estimate the number of protons contained in the Earth from the mass of the Earth and the mass of the nucleon (explain why the electron mass can be neglected in this calculation), knowing that most atomic nuclei have on average the same amount of protons and neutrons.

d) What would be the electrical charge of the Earth if all its electrons were stripped from it? What about that of the Moon?

e) Estimate an OM (order of magnitude) of the intensity of the electric force between the Earth and the Moon if their charges were as calculated in (d). Compare it with that of the gravitational force.

f) Why, on an astrophysical scale, is the electrical interaction negligible compared with the gravitational interaction?

[3] This atomic model is outdated because electrons do not have precise velocities and positions in the atom. However, it provides a good estimate of the order of magnitude of the forces involved.

Exercise 9: Mass and energy

The mass of bodies is one of many possible forms of energy (mechanical, chemical, radiation, heat). Appendix C gives the formula for the total energy of a particle moving at velocity v according to special relativity:

$$E_{\text{rel}} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

a) Calculate the mass energy of a 50 kg student. Compare it with his or her kinetic energy when traveling by airplane at a speed of 1000 km/h.

b) Show that in the limiting case where $v \ll c$ (so $x = v^2/c^2 \to 0$), the above relativistic formula can be written as the sum of the energy of the rest mass plus the Newtonian kinetic energy term:

$$E_{\text{rel}} \cong m\, c^2 + \frac{1}{2} m\, v^2 .$$

To this end, take advantage of the fact that for the function

$$f(x) = \frac{mc^2}{\sqrt{1 - x}}$$

and for $x \to 0$, we can use the approximation

$$f(x) = f(0) + f'(0) \cdot x .$$

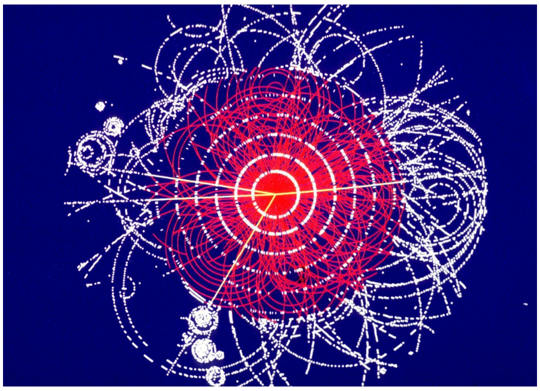
c) What must the student's velocity at point (a) be for his kinetic energy to be equal to his or her mass energy?

Exercise 10: The Higgs boson

The equivalence of matter and energy allows us to create mass from the kinetic energy of particles. At CERN, the Large Hadron Collider (LHC) accelerates protons and then collides them, turning their kinetic energy into mass energy. This allows us to produce, for short periods of time, particles that have never been observed before because of their very short lifetimes. These particles, however, existed in the primordial universe when it was denser and therefore more energetic.

Among these particles, the Higgs boson is the one that explains, among other things, what the origin of the mass of all particles is:
https://archive.nytimes.com/www.nytimes.com/interactive/2013/10/08/science/the-higgs-boson.html?_r=1&#/?g=true

Its discovery was first announced at CERN on July 4, 2012, after decades of research: its existence had been predicted theoretically using the mathematical models and unification arguments of Peter Higgs in 1964. For this discovery, P. Higgs and F. Englert were awarded the Nobel Prize in 2013. The image on the right shows the result of the collision between two protons that produced a Higgs boson that disintegrated into 4 muons (yellow tracks). Its mass is about $m_H \approx 2 \cdot 10^{-25}$ kg.



Credit: https://cds.cern.ch/record/765532

a) What is the rest mass energy of the Higgs boson (see previous exercise and Appendix C)?

b) If you want to produce a Higgs boson at rest from the collision between two protons, what is the minimum energy that each proton must provide?

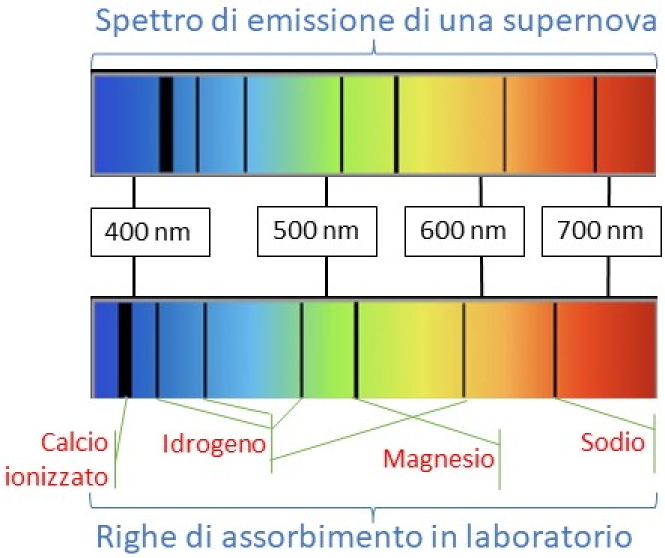
c) Determine the minimum velocity of protons for their collision to produce a Higgs boson.

   Warning: since the proton velocity approaches that of light, $v \sim c$, we cannot use Newtonian mechanics: to express the total energy of the proton, we must use the relativistic energy formula (Appendix C).

2 Expansion

Exercise 1: The supernova

The first band in the figure below represents the emission spectrum of a supernova: The black vertical stripes represent λ wavelengths that we do not receive because of their absorption by calcium, hydrogen, magnesium, and sodium atoms in or near the source. We know what wavelengths these absorptions correspond to in our laboratories (second strip). They are shifted relative to the observed values.



a) Using the scale in the figure above, determine the difference between the observed wavelength (λ_0) and the wavelength predicted (λ) for the magnesium absorption line. Deduce the redshift of the supernova.

b) What is the recession velocity of the supernova relative to us? Give the result in SI units as well as the ratio of this speed to the speed of light c.

c) Assuming that this shift is due solely to the expansion of the universe, provide an estimate of the supernova distance in Mpc using the Hubble-Lemaître law.

d) Derive an estimate of how long the light in this image traveled to reach us.

Exercise 2: The redshift

Consider a distant source emitting radiation with wavelength λ.

a) What is its redshift if today we observe this radiation with a wavelength: $\lambda_0 = 2\lambda$?

b) The same question as before, but if we observe $\lambda_0 = 3\lambda$ e $\lambda_0 = 10\lambda$?

c) What must the redshift of a source be for a light emitted in green to be observed in red?

Exercise 3: The formula of the Doppler effect

a) Prove that the formula relating the redshift of a source $z = (\lambda_0 - \lambda)/\lambda$ to its recession velocity v with respect to the observer is equivalent to the non-relativistic Doppler effect equation for a wave propagating at velocity c:

$$f_0 = f \cdot \frac{c}{c+v} \ .$$

b) For what values of z is this formula valid (i.e., for what values can the cosmological redshift be likened to a standard Doppler effect)?

Exercise 4: Non-relativistic limit of the relativistic Doppler effect

The relativistic Doppler effect is given by the formula

$$1 + z = \frac{\lambda_0}{\lambda} = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} \tag{1}$$

where $\beta = v/c$, λ_0 is the observed wavelength and λ is the emission wavelength of a moving source.

a) What is the limit of z when $\beta \to 1$ ($v \to c$)?

b) Prove that, in the nonrelativistic limit $\beta \to 0$ ($v \ll c$), formula (1) above becomes the classical formula for the Doppler effect:

$$\frac{\lambda_0}{\lambda} = 1 + z = 1 + \beta \ . \tag{2}$$

Exercise 5: The Hubble constant

With only one significant digit, current estimates of the Hubble constant indicate.

$$H_0 \approx 7 \cdot 10^4 \, \frac{\text{m/s}}{\text{Mpc}} \, .$$

a) Convert this value to $\frac{\text{nm/year}}{\text{km}}$. Can we feel the effects of this expansion at our scale?

b) What is the expansion velocity at the diameter scale of the solar system ? Compare its order of magnitude (OM) with that of the velocity of Pluto around the Sun v_P = 4.74 km/s (= 17'100 km/h).

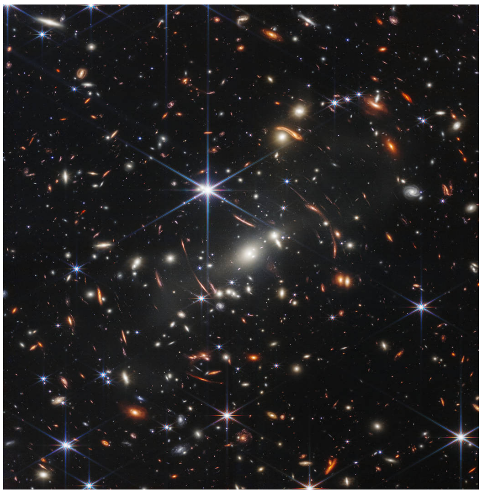
c) What is the expansion velocity at the diameter scale of our galaxy? Compare its OM with that of the Sun's rotational velocity around the galactic center $v_\odot = 220$ km/s.

d) What is the velocity of expansion at the scale of the average size of galaxy groups? Compare it with the velocity at which Andromeda (image below) is approaching our galaxy $v_A = 111$ km/s.

Credit: https://commons.wikimedia.org/wiki/Halpha_sharp.jpg, David Dayag

Exercise 6: First *deep field* of JWST

The following image is from July 2022 and represents the visible reconstruction of the first deep field of the James Webb Space Telescope (JWST), designed to detect electromagnetic waves in the infrared. It is a tiny portion of the sky, with a side of about one arcsecond, but rich in galaxies at different distances. The bright stars are those belonging to our galaxy. The image is centered on the cluster SMACS 0723, whose two brightest elliptical galaxies are clearly visible in the center of the image. The redshift of this cluster is $z = 0.388$.



Credit: NASA, ESA, CSA, and STScI

https://www.nasa.gov/image-feature/goddard/2022/nasa-s-webb-delivers-deepest-infrared-image-of-universe-yet

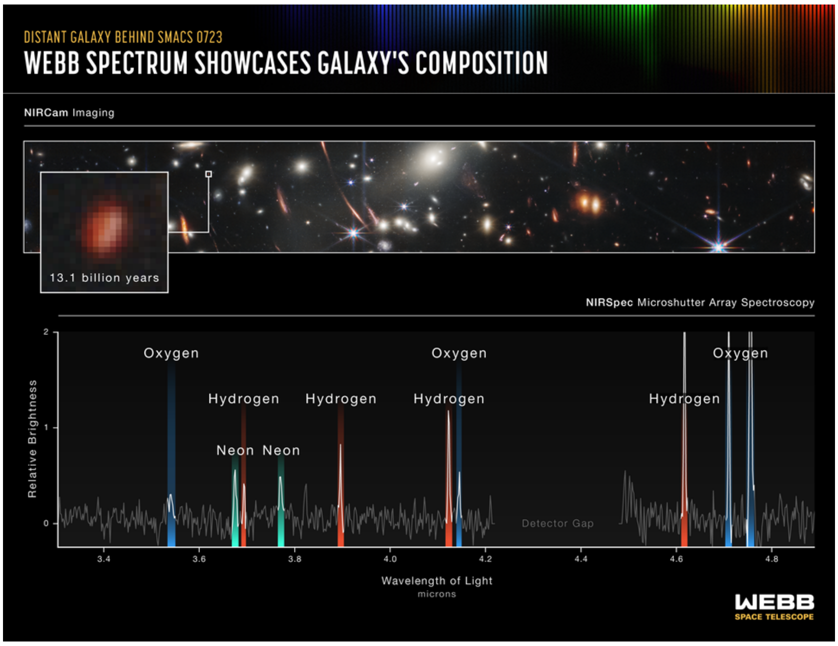
a) From the redshift, calculate the wavelength λ_0 detected by the telescope of the OIII oxygen emission line of SMACS 0723, knowing that, for a laboratory-observed OIII source, this wavelength is $\lambda = 0.5007\,\mu$m.

b) Determine the distance to SMACS 0723. Is the expansion negligible at this distance? Justify your answer.

The SMACS 0723 cluster creates a gravitational lensing effect (discussed in Chapter 4) on many of the most distant sources, which consequently appear distorted and amplified[4]. In particular, the source shown in the enlargement below is among the most distorted. Also shown for this source are the detected emission wavelengths of some chemical elements, in μm. In particular, the last oxygen line (blue line on the far right) represents oxygen OIII[5].

[4]For an in-depth analysis see reference https://arxiv.org/pdf/2207.07101.pdf.

[5]The lines represent, from left to right, the OII doublet ($\lambda = 372.6$ and 372.8 nm), NeIII ($\lambda = 386.9$



Credit: NASA, ESA, CSA, and STScI https://esawebb.org/news/weic2209/.

c) By reading the observed wavelength value for the OIII oxygen line, emitted about 13.1 billion years ago from this distant source, calculate its redshift.

d) Verify that the answer found in (c) is consistent with the light travel time shown in the figure by comparing the result with the values given in the table on the following website: http://lcogt.net/spacebook/redshift.

e) Is this source located within the Hubble radius? Why or why not? Explain your answer using the formulas.

f) Can we calculate the distance to this source using the Hubble-Lemaître law? Explain.

g) Does the velocity at which this source moves away exceed the speed of light? If not, calculate this velocity. If so, explain why its light can still reach us.

nm), He ($\lambda = 388.9$ nm), NeIII ($\lambda = 396.8$ nm), H_ϵ ($\lambda = 397.0$ nm), H_δ ($\lambda = 410.2$ nm), H_γ ($\lambda = 434.05$ nm), the first OIII ($\lambda = 436.3$ nm), H_β ($\lambda = 486.13$ nm) and the other two OIII ($\lambda = 495.9$ and 500.7 nm).

Exercise 7: Twin universes

Read the comic book "Cosmic Story" :
http://www.savoir-sans-frontieres.com/JPP/telechargeables/English/cosmic_story_anglo_indien/cosmicstory_eng-1.pdf/

In light of your knowledge of the Big Bang, what is the problem with the "bipolar universe" model (pages 57-60 of the comic book) ?

Exercise 8: Redshift of the Cosmic Microwave Background (CMB) radiation

a) Knowing that the universe at the time of decoupling had a temperature $T = 3000$ K, determine the wavelength of the peak of the emission spectrum λ_{max} of the CMB at that time. Use Wien's law (Appendix D.1 on blackbody radiation).

b) We know that today the CMB has a temperature of about 2.7 K. What is its redshift?

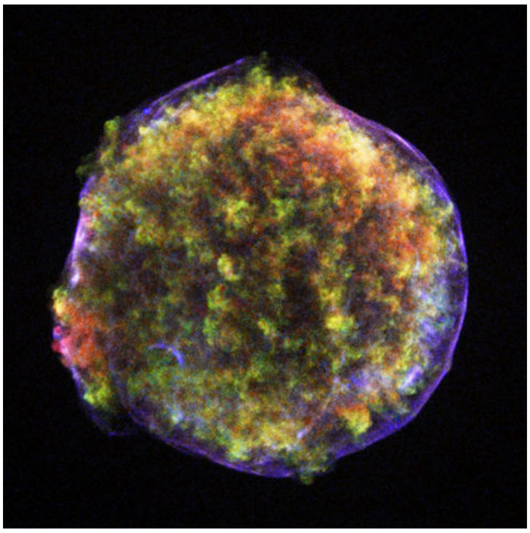
c) Using the Hubble-Lemaître law, can we determine the distance that separates us from decoupling? If so, try to do so. If not, explain why.

Exercise 9: Luminosity distances

In exercise 1 we saw that it is possible to determine the distance of some sources from its redshift.

However, for some sources called **standard candles**, of which we know the absolute luminosity L (the radiative power, in W) we can determine the distance by measuring the received flux f (the power per unit area, in W/m^2). The distance determined in this way is called the **luminosity distance**, denoted by D_L (Appendix D.2).

An important example of a standard candle used in cosmology are the supernovae Ia explosions.



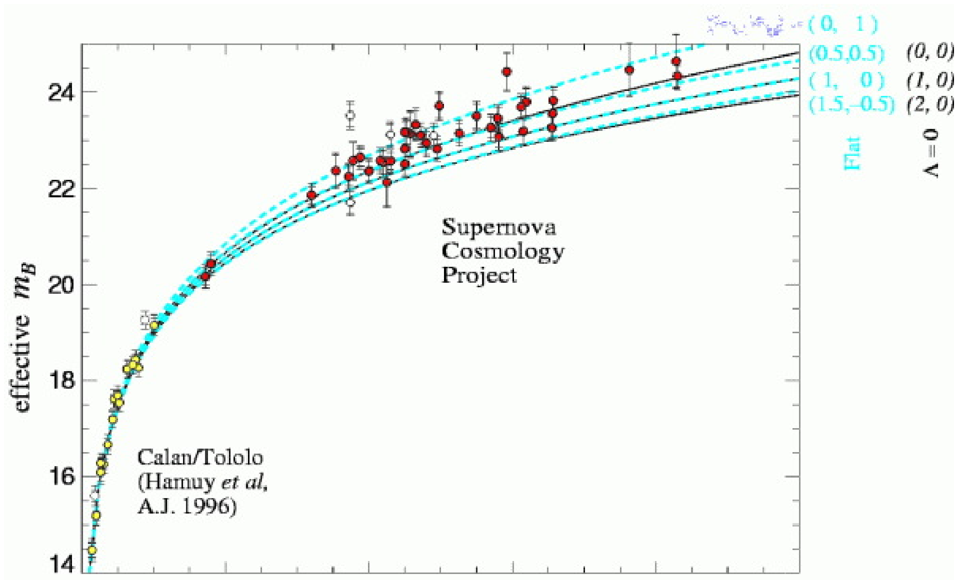
Credit: NASA/CXC/Rutgers/J.Warren & J.Hughes et al. This photo is an X-ray image of the remains of Supernova Ia observed in 1572 by the Danish astronomer Tycho Brahe.

Video of the Supernovae Ia at the address:
https://en.wikipedia.org/wiki/Type_Ia_supernovaa

a) State the formula that expresses the distance D_L between an observer and a source as a function of its absolute brightness L and the measured flux f. Assume that the emission from the source is spherically symmetric.

b) Knowing that the luminosity of a supernova Ia is on the order of ten billion times higher than that of the Sun ($L_{\text{Sun}} = 4 \cdot 10^{26}$ W), what is the OM of its distance if we measure a flux of 10^{-13} W/m^2? Provide the answer in m and Mpc.

c) By measuring their luminosity, and comparing it with redshift, it is observed that, for the most distant sources, the corresponding distances are much greater than those predicted if the universe were composed exclusively of matter – both ordinary and dark matter – as it would have to slow down its expansion by gravitational attraction (graph below). How can this fact be explained?



Credit: Perlmutter *et al.*, Astrophysical Journal 517, 565-586, 1999.

The graph above shows measurements published in a scientific paper in 1998 (Perlmutter et al., Nobel Prize in 2011).

This is the magnitude of supernovae Ia, m_s (which is a function of luminosity: $m_s = -2.5 \cdot \log \mathrm{f_s} + \mathrm{constant}$) as a function of redshift.

Exercise 10: True or false ?

Justify each response.

1. Spiral galaxies are older than elliptical galaxies.
2. Baryons are particles that interact only gravitationally.
3. Because of the expansion of the universe, the velocity of recession between two very distant galaxies can exceed the speed of light, even though special relativity states that nothing can go faster than c (Appendix C).
4. In special relativity, matter is a form of energy (Appendix C). So dark matter and dark energy are two different ways of defining the same thing.
5. Just before decoupling, dark matter was free to create over-density by gravitational attraction because it did not interact with electromagnetic radiation.
6. The Big Bang is an explosion that took place at a certain instant and gave birth to the universe.

Activity: The cosmological redshift

When we observe distant galaxies, the light we receive is systematically redshifted.

a) Explain why the waves we receive always have a wavelength λ_0 greater than the wavelength λ emitted by the source.

The following image shows an example of the emission spectrum of the same type of galaxy[6] at different distances. This is the intensity of radiation received (y axis, in W) for each wavelength (x axis, in nm). From bottom to top, we see:

1. The emission spectrum observed in our galaxy at a distance of $r = 0$ ly. This represents the spectrum as it is at emission, without any shift (λ).

The following spectra represent the same as the first, but received from increasingly distant galaxies: these spectra are increasingly redshifted. ($\lambda_0 > \lambda$):

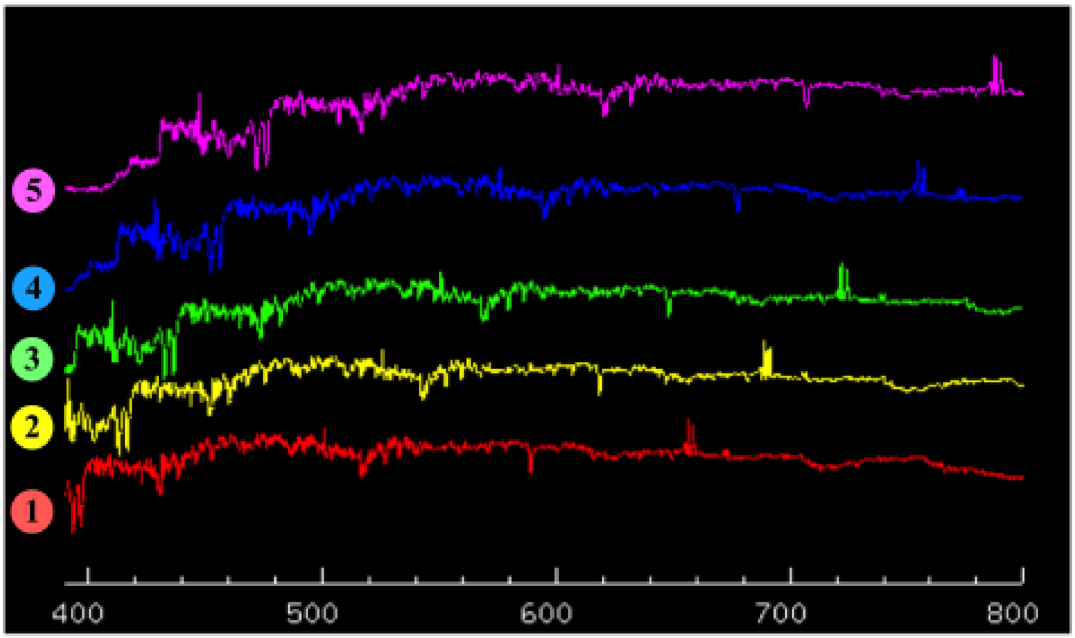
2. The same spectrum, received from a galaxy at the distance of $r = 6.0 \cdot 10^8$ ly;
3. The same spectrum, received from a galaxy at the distance of $r = 12 \cdot 10^8$ ly;
4. The same spectrum, received from a galaxy at the distance of $r = 18 \cdot 10^8$ ly;
5. The same spectrum, received from a galaxy at the distance of $r = 21 \cdot 10^8$ ly .

b) Using the most pronounced emission peak (the one for which λ is about 650 nm in our galaxy), complete the following table.

	λ_0 [nm]	$\lambda_0 - \lambda$ [nm]	$z = \frac{\lambda_0 - \lambda}{\lambda}$	r [Mpc]
Peak of spectrum 2)				
Peak of spectrum 3)				
Peak of spectrum 4)				
Peak of spectrum 5)				

c) How many significant digits have the least accurate data in this table ?

[6] These are different galaxies at different distances, but with the same chemical characteristics and therefore emitting the same spectrum.



d) Using the data from the table, plot the corresponding redshift points (y axis) as a function of distance from the galaxy (x axis) on a graph. *You can use a spreadsheet such as Calc or Excel. Do not forget to indicate the title, quantities and units on the axes.*

e) Can we see if there is a relationship between z and r? If yes, of what kind?

f) Insert a trend line into this graph: a line with equation $z = a \cdot r + b$, where $b = 0$ (where z is the ordinate and r the abscissa). Explain why b must be zero.

g) What relationship between z and r is represented by this line? What is the numerical value (with an appropriate number of significant digits and the correct unit) of the constant a in the equation $z = a \cdot r$? To what physical quantity does it correspond?

h) Use the Doppler effect formula linking z and v (for $v \ll c$) to find the relationship between the velocities of the observed galaxies v and their distance r, from the result of question (g): $v = \text{constant} \cdot r$.

i) What is the numerical value of this constant (with an appropriate number of significant digits and the correct unit)? How can this result be interpreted from a cosmological point of view?

3 Basics of General Relativity

Exercise 1: Electric Charge and Gravitational Charge

Suppose there are two bodies of mass and electric charge m_1, q_1 and m_2, q_2 respectively.

a) Each of these bodies is placed separately at a distance d from a third mass M that has no electric charge. Write, for each of the two bodies, the formula expressing the acceleration due to gravitational interaction with M.

b) Each of these bodies is placed separately at a distance d from a third charge Q of negligible mass. Write, for each of the two bodies, the formula that expresses the acceleration due to the electrical interaction with charge Q.

c) Does the acceleration depend on the characteristics of bodies 1 and 2 in case (a) and in case (b)? Explain why.

d) Can the force be repulsive in case (a)? and in case (b)? Explain why.

Exercise 2: The Crazy Plane

1. What can we say about the velocity of the plane in the picture below? What about its acceleration?

2. Without more information, can we know whether the plane is approaching or receding from the Earth's surface?

Credit: https://www.wired.com/2013/09/how-do-you-pour-water-up-into-a-glass/

Exercise 3: Inertial Motion?

For each of the following motions:

a) draw the two force diagrams on the body in question

b) specify whether the motion is inertial

   i) according to Newtonian mechanics (NM) and
   ii) according to general relativity (GR).

1. A probe traveling in space, away from any mass.
2. A comet moving away from the Sun in an elliptical orbit.
3. An apple on the ground.
4. The carriage of a train in uniform rectilinear motion.
5. The carriage of a train in uniformly accelerated rectilinear motion.
6. A bicyclist riding along a track in a uniform circular motion.
7. The Earth in its motion, considered circular and uniform around the Sun.
8. A hammer dropped with zero initial velocity on the surface of the Moon.
9. A raindrop falling on the earth's surface at a constant velocity.

Exercise 4

Choose the box corresponding to the correct statement, and then justify your choice. You can proceed by exclusion by explaining why some choices are wrong.

1. The equivalence between mass and energy implies

   ☐ that gravitational mass is different from inertial mass.
   ☐ that a body has energy just by the fact that it has mass.
   ☐ that mass is always conserved.
   ☐ that the shape of space-time is not affected by the energy it contains.

- ☐ that the shape of space-time is affected by the mass it contains only if the mass is large enough.

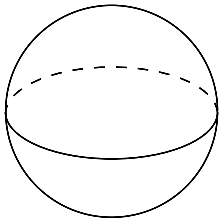
2. A person is enclosed in a windowless space shuttle, far from any stars. For a few minutes he is pressed against one side of the shuttle, then floats inside it. One possible explanation is that

   - ☐ the shuttle was first accelerated and then decelerated to a stop.
   - ☐ the shuttle was first accelerated, then the acceleration ceased and the shuttle continued to move at a constant speed.
   - ☐ the shuttle always moved at a constant speed.
   - ☐ the shuttle always moved with constant acceleration.
   - ☐ the shuttle first followed a straight and uniform motion and then, as the person floated, accelerated.

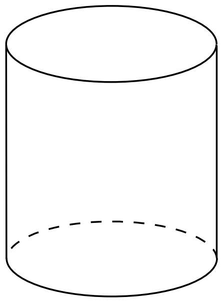
Exercise 5: Solids

a) Each of the following objects (or parts of objects) is associated with the solid next to it. On the surface of each solid, color

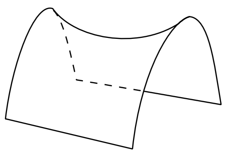
- in yellow the parts of surfaces, lines or points consisting of points with positive Gauss curvature k_G;
- in blue the parts of surfaces, lines or points consisting of points with negative k_G;
- in red the parts of surfaces, lines or points with null k_G.

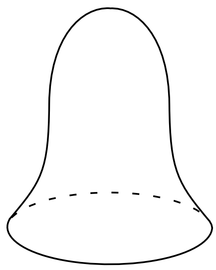
1. Sphere

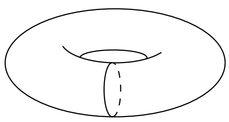
2. Cylinder

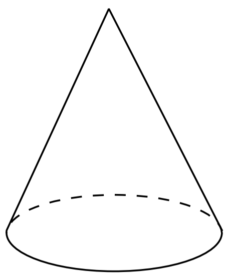
3. Saddle

4. Bell

5. Torus

6. Cone

b) For each colored solid, can we determine what the sign of the total curvature K is. Calculate it, if possible.

Exercise 6: Osculating Circle of a Parabola

Consider the parabola of equation $y_p(x) = x^2$.

a) Determine the radius of curvature R of the osculating circle of this parabola at the point (0;0). What is its curvature at this point?

b) Find the curvature of the same parabola, but at point (1;1).

c) Write the equation expressing the curvature at a generic point of the parabola $(x; y = x^2)$ as a function of the abscissa of point: $k(x)$.

d) Why does it not make sense to speak of an osculating circle of a straight line?

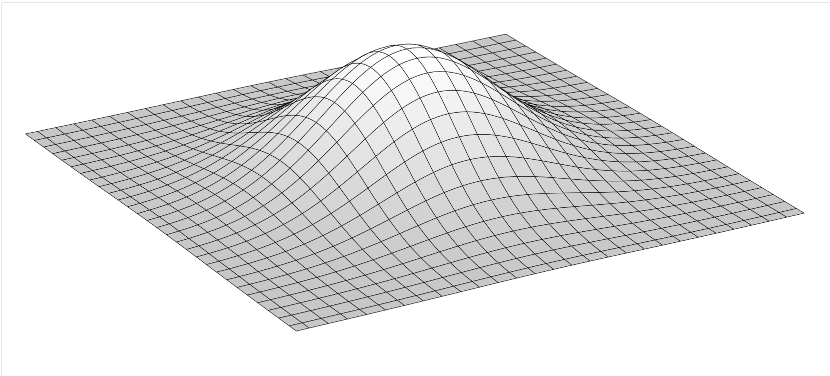
Recommended method: write down the generic equation of the circle and then explicate $y_c(x)$ (pay attention to the sign of the root). Then solve the system by imposing 1) passage through the chosen point, 2) equality of the prime derivatives $y'_c = y'_p$, and 3) equality of the second derivatives $y''_c = y''_p$.

Exercise 7: The Bump

Imagine that you are holding a thin, flexible, flat sheet of paper. Press lightly with one finger from underneath pushing upward until you create a bump like the one in the figure below.

a) What is the Gauss curvature of the points on the sheet before pressing? What about the total curvature of the sheet?

b) Answer the same questions, but for the warped paper.



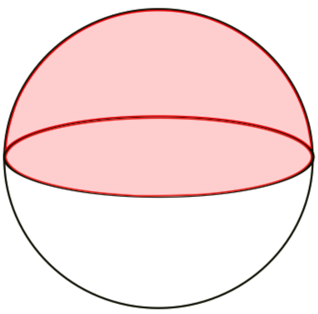
Exercise 8: Parallel Transport

a) Using the definition of total curvature of a surface, determine what the total curvature of the surface is.

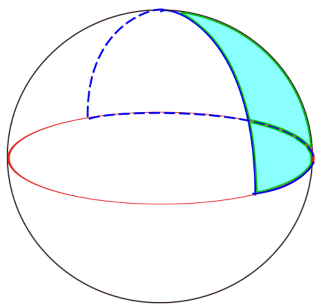
1. of a sphere;
2. of half a sphere (part A of the figure below);
3. of a quarter sphere (part B of the figure below);
4. of an eighth of a sphere (parts C and D of the figure below).

b) Use the parallel transport method to test the 4 results of the previous question. For the eighth sphere, test both possibilities: dividing the quarter in the two possible directions so that you get a triangle or a "biangle" (parts C and D of the figure below).

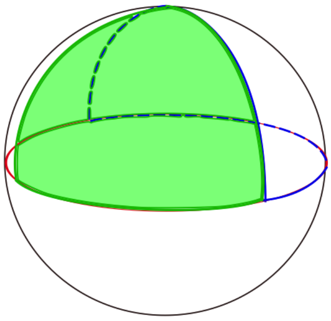
c) Generalize the results given in (a): what is the total curvature of the surface of the n^{th} part of a sphere?



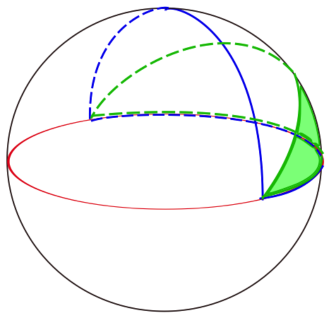
A



B



C



D

Exercise 9: The Cylinder

a) On a sheet of A4 paper, place two points P and Q so that their distance is the same order of magnitude as the sheet. Draw the geodesic that passes through these two points. Does it correspond to the shortest path between these two points?

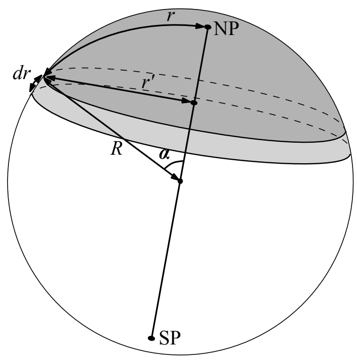
b) Using the same sheet of paper, construct a cylinder by gluing two opposite edges together so that the geodesic drawn in (a) is on the outside of the cylinder thus created.

1. Is the geodesic drawn at point (a) still a geodesic for the cylinder? Is it still the shortest path between P and Q? If not, plot the shortest path on the cylinder.
2. How many geodesics connecting P and Q exist on the cylinder surface?
3. What happens if instead of drawing the points P and Q as shown in (a), we draw them very close together, in the center of the paper? What can we infer from this?

Exercise 10: Circles, Discs and Spheres in a Curved Space

In a flat two-dimensional space, the formulas for the perimeter and area of a disk of radius r are $P(r) = 2\pi r$ and $A(r) = \pi r^2$.

a) Using the figure opposite, write the equations corresponding to those shown above, but in a curved space, with constant positive Gauss curvature $k = 1/R^2$ (the sphere in the drawing): $P(r, R)$ and $A(r, R)$.



b) For the two formulas found in Step 1, check that for the limiting values:

- $r = 0$ (at the north pole, NP),
- $r = \pi R/2$ (at the equator),
- $r = \pi R$ (at the south pole, SP)

expected results are found, for example, for the perimeter

$$P(r = 0) = 0\ , \quad P(r = \pi R/2) = 2\pi R \quad \text{and} \quad P(r = \pi R) = 0.$$

c) Following the same procedure for the area and perimeter of a circle of radius r, write the formula for the volume of a sphere of radius r in a curved 3D space with a positive constant radius of curvature.

Recommended method: consider a uniformly curved space (radius of curvature R). Just as the (2D) area of a circle of radius r corresponds to that of the canopy of radius r on the 3D sphere of radius R, the volume of a (3D) sphere of radius r corresponds to the volume of the canopy of the 4D hypersphere of radius R. For this point, it is not possible to visualize it . One must trust the calculations – and test the result with the limiting case $r \to 0$.

The following links give access to two comics that give us insight into what happens with geometric quantities in a curved space.

The Black Hole:
http://www.savoir-sans-frontieres.com/JPP/telechargeables/English/THE_BLACK_HOLE.pdf

All comics by Jean-Pierre Petit translated in English:
http://www.savoir-sans-frontieres.com/JPP/telechargeables/free_downloads.html#english

Exercise 11: Two Visions of Free Fall

a) State the *Principle of Equivalence* in its two versions.

b) Explain what inertial motion is (i) in Einstein's theory and (ii) in Newton's theory.

c) For each situation, choose the box corresponding to the theory for which it is an inertial motion.

1. A shuttle traveling at constant speed in space, away from any celestial body.

   ☐ Newtonian mechanics ☐ general relativity

2. A skydiver at the beginning of his fall, with a constant acceleration equal to g.

   ☐ Newtonian mechanics ☐ General relativity

3. The skydiver himself at the end of the fall when he reaches his constant limit velocity.

   ☐ Newtonian mechanics ☐ General relativity

4. An immobile locomotive on the Earth's surface.

   ☐ Newtonian mechanics ☐ General relativity

5. An exoplanet orbiting its star.

   ☐ Newtonian mechanics ☐ General relativity

6. A projectile following a parabolic motion, launched from the surface of a planet.

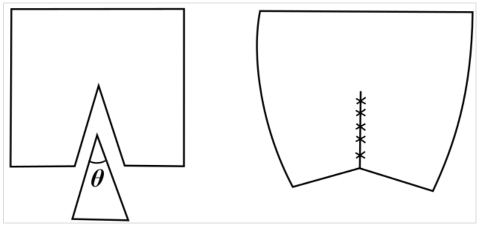
   ☐ Newtonian mechanics ☐ General relativity

Activity: The Cone Curvature

A cone is a solid with zero Gaussian curvature everywhere except at the vertex.

a) Using a flat sheet of paper on which you have drawn a grid of geodesics (straight lines), construct a cone as follows:

1. Draw an arbitrary angle θ in the center of the paper and measure it (in degrees, then convert to radians).
2. Cut out the angle with scissors.
3. Glue the two cut edges of the sheet (without overlapping them).



b) What can be said about parallel geodesics on the cone? Do they converge, diverge or remain parallel? *Remember that the geodesics must have the same angle of intersection with the original grid.*

c) Following the geodesics on the surface of the resulting cone, draw a triangle ABC so that the vertex of the cone is inside it.

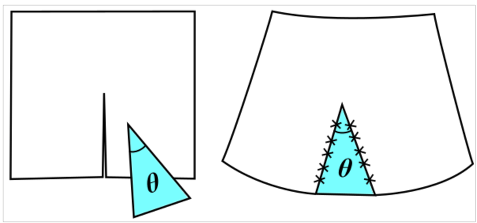
d) Using the method of parallel transport, measure the total curvature K of the surface of the drawn triangle. In your opinion, is there a relationship between K and θ?

e) Measure the interior angles of triangle ABC (call them α, β and γ) and calculate their sum. What is the relationship between $\alpha + \beta + \gamma$ and the angle θ at point (a)? Prove this relationship geometrically.

f) Using the results found in (d) and (e), provide the formula that relates the sum of the interior angles of a triangle to its total curvature, then compare it with the formula provided in the course (Chapter 4).

g) How do the answers to (b), (c), (d), (e), and (f) change if we add an angle θ to (a) instead of removing it?

h) Explain why the Gaussian curvature of the vertex of a cone is infinite.



4 Gravitational Lenses

Exercise 1: Deflection angle α (dimensional analysis)

We can find a formula for the angle of deviation without the numerical factor (this factor is 2 for Newton and 4 for Einstein), but in a simpler way: with the help of dimensional analysis. The idea is to consider that this deviation should depend only on 3 quantities:

1. Gravitational acceleration at the nearest distance from the gravitational lens $g = \frac{GM}{d^2}$. Being an acceleration, its fundamental unit is $\mathrm{m \cdot s^{-2}}$
2. The impact parameter d (passage distance), whose SI unit is the meter (m) ;
3. The passage velocity c, whose SI unit is $\mathrm{m \cdot s^{-1}}$;

Since the deviation is an angle, without dimensions, its SI units are radians. So in a combination $\alpha \sim g^p \cdot d^q \cdot c^r$, where p, q and r are integers, the meters and seconds must necessarily simplify. Use this idea to find the constraints for the integer exponents p, q and r. Choose the simplest solution to find the formula (without a numerical factor) for the deviation.

Exercise 2: Deviation from the Sun

According to the relativistic formula, obtained in 1915 by A. Einstein, the deflection of a ray of light is twice that obtained by an approximate Newtonian calculation.

Experimentally, we can measure this deviation of light from a distant star as it passes close to the Sun. The idea is to measure the position of a star in the sky when the Sun is not present, and then compare this position with the position measured when the Sun's image is close to that of the star: the image will be shifted by an angle α. However, to do this, it is necessary to make this second measurement during a total solar eclipse, otherwise the daylight would not allow you to see that of the star.

Calculate the deflection of a ray of light passing at a distance $d = R_{\mathrm{Sun}}$ from the center of the Sun using

1. the newtonian formula,
2. the relativistic formula

Historical note

Eddington's observation of the deflection angle of starlight during the solar eclipse of 1919 was the first experiment that confirmed Einstein's theory. In reality, Eddington's

measurement alone was not accurate enough to solidly confirm the result of general relativity, and a definitive consensus was formed only during the last century.
Link ESA :
https://www.esa.int/Science_Exploration/Space_Science/Relativity_and_the_1919_eclipse

Exercise 3: Lunar Lensing ?

A deviation of the order of $1''$ of a source image can be easily measured.

a) Convert $1''$ into radians.

b) Calculate the impact parameter d that a beam of light must have for the Moon to produce a deflection $\alpha \approx 1''$.

c) Why can't the Moon produce observable gravitational lenses? What should be the order of magnitude of its density to produce observable lenses?

Exercise 4: RCS2 032727-132623

The following image was taken by the Hubble Space Telescope in 2012. In bright yellow, in the center, we see the galaxy cluster RCS2 032727-132623, about 2 Gpc away from us.

a) What are the clues to indicate that this cluster is a gravitational lens? Explain your answer with a diagram showing the path of light rays to the observer.

b) What kind of gravitational lenses are these? Justify your answer.

Credit: NASA, ESA, J. Rigby, K. Sharon, M. Gladders and E. Wuyts, http://www.space.com/14481-hubble-photo-brightest-galaxy-gravitational-lens.html.

c) Why is an Einstein ring not observed in this case?

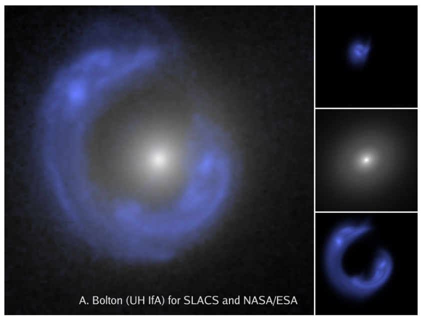
d) Can we distinguish images from the same source?

Exercise 5: Einstein's ring

The figure below shows one of the first Einstein rings ever observed. It dates from 1998 and shows the galaxy SDSSJ1430 (in blue) seen through a gravitational lens. On the left, in white in the center, we see the much closer galaxy that acts as a lens.

The images on the right, from top to bottom, represent respectively:

1. a computer reconstruction of how the source image should look without a lens,
2. the lens alone and
3. the distorted image alone.



Credit: http://apod.nasa.gov/apod/ap080728.html.

a) Explain under what conditions we can observe these kinds of images.

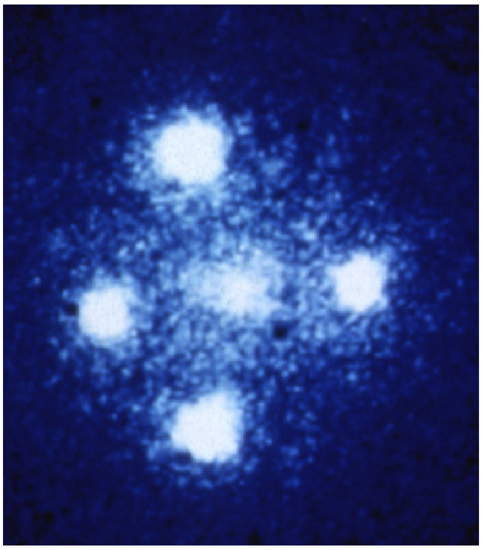
b) Astrophysicists use these observations to estimate the mass of the lens: they measure the redshifts for the lens $z_L = 0.285$ and for the source $z_S = 0.575$, as well as the Einstein radius $\theta = 1.51''$. Using these data, calculate the mass of the lens. Give the result in kg and solar masses $M_\odot$. Do not forget to convert θ to radians.

c) How many significant digits does this result have? What are the sources of uncertainty?

d) Why is this type of large-scale mass estimation important in modern cosmology?

Exercise 6: Einstein's cross

This photo of Einstein's cross dates back to 1990 and is the most detailed photo ever taken of this object. The 4 images forming the cross have a redshift of 1.7, which corresponds to the distance of 3 Gpc, while the galaxy at the center has a redshift of 0.0394. In addition, the average angular distance between the four images of the cross and the center was measured. We can consider this angular distance as a good estimate of the associated Einstein radius, which is 0.8 arcsecond.



Credit: NASA, ESA, and STScl

a) Explain (1) what the phenomenon behind this observation is, (2) what the conditions for this cross-shaped image to occur are.

b) Using Hubble's Law, verify that the distance of the galaxy from the center of the image is 0.2 Gpc.

c) From this, deduce the distance between the galaxy at the center (L) and the source of the 4 images (S).

d) Determine the Einstein radius of this image in radians.

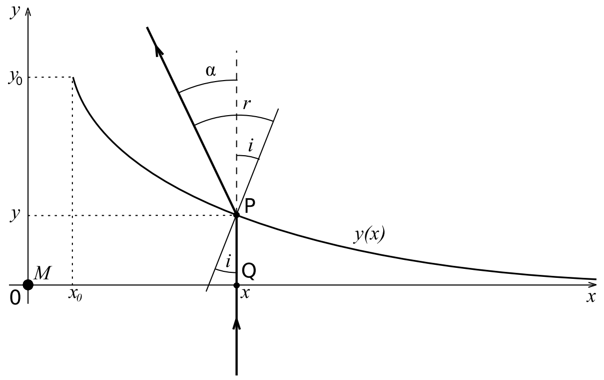
e) Write the formula that relates this Einstein radius to the mass of the galaxy at the center of the image.

f) Derive an estimate of the mass of the galaxy at the center of the image in kg, then in $M_\odot$.

Exercise 7: Why a Wine Glass?

a) Strong lenses can be simulated using an optical lens (such as Plexiglas), provided its profile is well chosen. Explain why a biconvex profile is not suitable.

Consider an optical lens of refractive index n, with a shape similar to that of the base of a wine glass. We want to know what its profile should be so as to simulate a point gravitational lens of mass M, placed at the origin of the coordinate system. The graph below left represents the desired lens: its base is flat (the x-axis) and the profile is given by the unknown function $y(x)$ (photo at right).



b) Imagine a ray of light incident perpendicular to the base of the lens at a generic point Q far enough from the center, with coordinates $(x; 0)$. That ray will not be deflected at this point. Why?

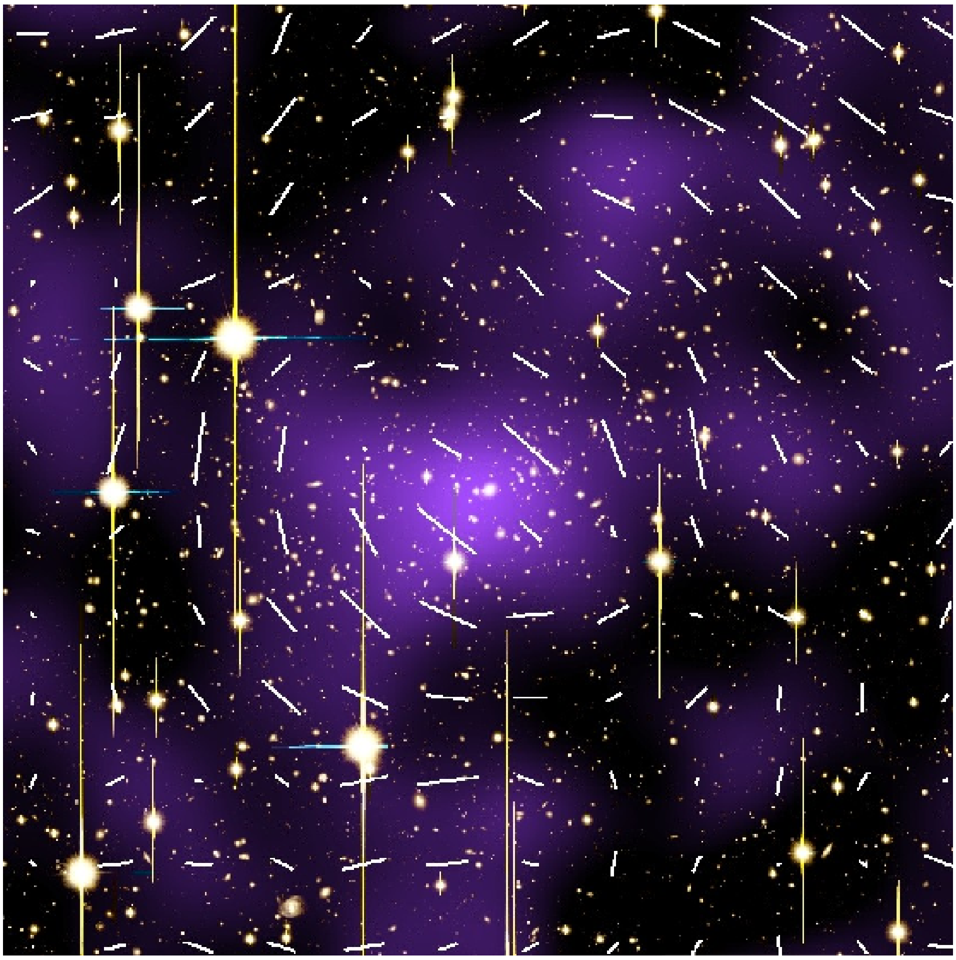
c) On the other hand, it will be deflected at point $P(x; y(x))$, at the exit of the glass. Write the law of refraction connecting the angles r, i and the refractive index n, with the approximation $\sin(r) \cong r$ and $\sin(i) \cong i$ (since $r \ll i$ and $i \ll 1$).

d) Knowing that $r = i + \alpha$, where $\alpha = 4GM/c^2x$ is the deviation we wish to have for the light ray (this deviation is *inversely* proportional to the passage distance between the light ray and the mass M), and using the equation found in (c), write the relationship between i, n and α, then write i as a function of G, M, c, x and n. Since G, M, c and n are constants, we find how the angle of incidence should vary as a function of $x : i(x)$.

e) Since i is the angle between the incident (vertical) ray and the normal to the curve $y(x)$ at point P, the tangent to this curve at this point is $y'(x) = dy/dx = -i(x)$. Substitute the expression obtained in (d) for $i(x)$ and then integrate this equation to find the profile $y(x)$.

Exercise 8: A2390

This is an image of the galaxy cluster A2390. The angular size of this image is 12' × 12', and in each cell (1' × 1') lines have been added.



Credit: Oguri, Masamune et al. MNRAS. 405 (2010) 2215-2230 arXiv:1004.4214 [astro-ph.CO] (https://inspirehep.net/record/853072/plots)

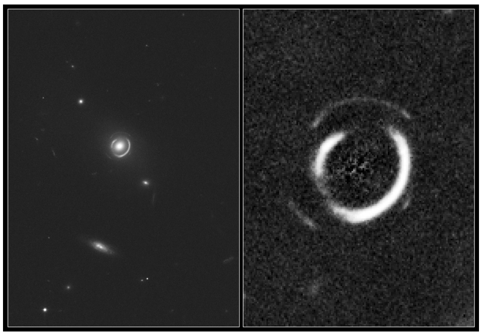
a) Can we say that there is a gravitational lensing effect in this image? If yes, what kind of lensing is it? In each case, justify your answer.

b) What do the added lines in each cell of the image represent?

c) What do the purple-colored (lighter) areas represent?

Exercise 9: Double Ring

The image below shows a unique recently observed phenomenon[7] from the Hubble satellite. These are two concentric rings around the same lens, consisting of a dwarf galaxy with redshift $z_L = 0.222$ (the light source in the center of the image on the left). The Einstein radius of the first ring is $\theta_1 = 1.43''$ and its redshift is $z_1 = 0.609$. For the second, less visible ring, we measured $\theta_2 = 2.07''$ and $z_2 \approx 3$.



Credit: https://esahubble.org/news/heic0803/ and https://en.wikipedia.org/wiki/SDSSJ0946%2B1006

a) Answer the following three questions about this image:

- What type of gravitational lensing effect underlies this observation and why?
- What conditions allow the formation of this ring image? Precisely draw the trajectory of the rays of the sources.
- The source of the second ring would not be observable without the lensing effect. Why?

b) From the redshifts of the lens and the first ring, calculate the distances

1. between the observer and the lens (D_{LO});
2. between the observer and the source of the first ring (D_{SO});
3. infer the distance between the lens and the source of the first ring (D_{SL}).

c) Using the data for the first ring, calculate the mass of the lens in kg, then in $M_\odot$.

d) What kind of material is the mass found at point (c) made of?

e) What other type of gravitational lensing is used to determine the distribution of non-luminous matter in the space between galaxies?

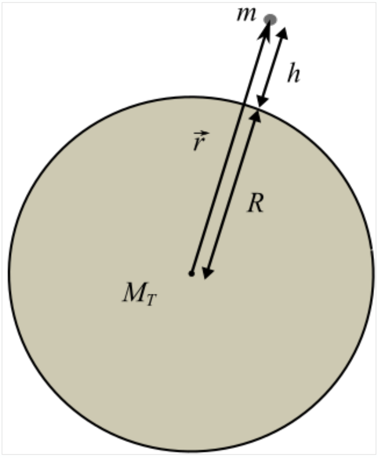
[7] R. Gavazzi *et al.*, *The Sloan Lens ACS Survey. VI: Discovery and analysis of a double Einstein ring* (2008), https://arxiv.org/abs/0801.1555.

5 Black Holes

Exercise 1: Gravitational potential energy

In the first two years of high school, you learned that the gravitational potential energy of an object on the Earth's surface is $E_g(h) - E_g(0) = E_g(r) - E_g(R) = mgh$ where h is its height above the Earth's surface ($h = r - R$, where r is the distance from the center of the Earth and R is the radius of the Earth) and m is its mass.

This formula is valid only if h is negligible with respect to the radius of the Earth. In this way $g = GM_T/r^2$ can be considered a constant, in fact:



$$r = R + h \cong R \qquad \Rightarrow \qquad g \cong GM_T/R^2 \,.$$

Otherwise, one should use the more general expression for gravitational potential energy, which is (see course)

$$E_g(r) = -\frac{GM_T m}{r} \,.$$

Prove that, if $h \ll R \Rightarrow r = R+h \cong R$, then $E_g(r) - E_g(R) \cong mgh$, where $g = GM_T/R^2$ (according to the definition of Earth's acceleration of gravity).

Hint: develop the difference algebraically:

$$E_g(r) - E_g(R) = -\frac{GM_T m}{r} + \frac{GM_T m}{R} = -\frac{GM_T m}{R+h} + \frac{GM_T m}{R}$$

$$= \frac{\dots}{(R+h)\cdot R} = \dots \cong mgh \,.$$

Exercise 2: Escape velocity

Calculate the escape velocity for an object thrown from the:

1. surface of the Earth,
2. surface of Mars,
3. surface of the Moon.

Exercise 3: The chemical composition of the atmosphere of planets

The temperature (in K) of a gas is directly proportional to the average kinetic energy of its particles. If we neglect the vibrational and rotational motions of the particles, this proportionality translates into the relationship

$$E_{\text{cin}} = \frac{3}{2} k_B T$$

where k_B is the Boltzmann constant : $k_B = 1.38 \cdot 10^{-23}$ J/K (Appendix E).

a) Using the formula for the kinetic energy of a particle of mass m and velocity v and the proportionality relation between E_{cin} and T, express the average velocity of the particles of a gas as a function of the temperature of the gas and the mass of the particle: $v_T(m; T)$. This velocity is called the ***thermal velocity*** of the gas.

b) Calculate the value of the thermal velocity of oxygen at the average temperature at the earth's surface: 15 °C, called $v_T(O_2; 15°\text{C})$. Is this velocity higher or lower for hydrogen H_2? And for nitrogen N_2? *For nitrogen and hydrogen, answer without performing calculations.*

c) Compare this velocity with the Earth's escape velocity, calculating the ratio

$$\eta_{\text{Earth}} = \frac{v_T(O_2; 15\ °\text{C})}{v_{\text{escape Earth}}}.$$

d) Repeat the same calculations as in (b) and (c) but for the Moon at the time of its formation, knowing that the average temperature on the Moon at that time (4 billion years ago) was about 2000 °C (1 significant digit).

e) Explain why the Moon has no atmosphere.

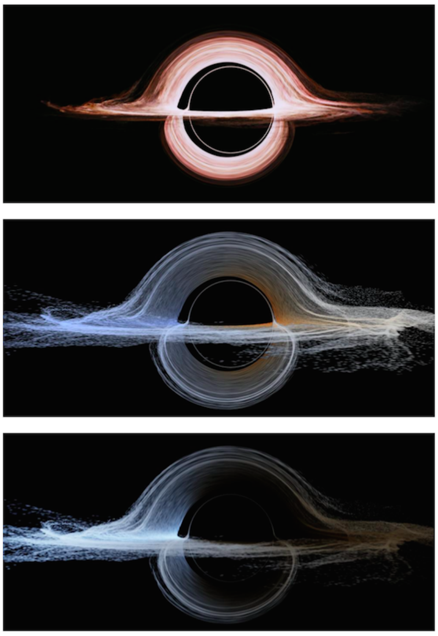
Useful link for future exercises: table of OM for densities
https://en.wikipedia.org/wiki/Orders_of_magnitude_(density)

Video:

- Simulation of the collision between two black holes:
  http://apod.nasa.gov/apod/ap151020.html
- A star swallowed by a black hole:
  http://apod.nasa.gov/apod/ap151028.html

Exercise 4: The Schwarzschild radius

The figure below shows Gargantua, the black hole in the movie "Interstellar": as it appears in the movie (top image) and according to the result of numerical simulation based on the equations of general relativity (the two bottom images). In all three images, the black hole is rotating (counterclockwise when viewed from above), and its accretion disk, composed of radiating matter, which plummets toward the black hole and also rotates at high speed.



Credit: O. James *et al.*, Class. Quantum Grav. 32 065001 (2015).

a) Indicate what the Schwarzschild radius is in each of the Gargantua images and explain why the black hole image is enveloped below and above by a bright disk.

b) Explain why in the most realistic images the accretion disk shows shades of blue or red.

c) Calculate the Schwarzschild radius of Gargantua, whose mass is 100 million times that of the Sun.

d) Make the same calculations as in (c) but for a black hole of the mass of the Sun, then the mass of the Earth.

e) What would be the density of Earth if all of its mass were concentrated in its Schwarzschild radius? Are there any known objects with this density?

Exercise 5: The supermassive black hole in Sgr A*

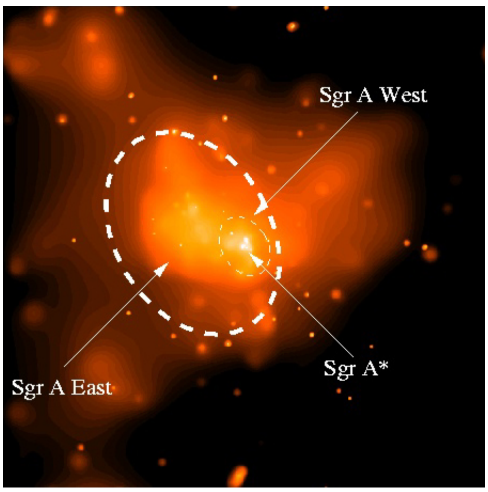
Suppose that S2 has a uniform circular motion (UCM) of radius $r \approx 1000$ AU – 10 times greater than its perihelion – and that its scalar velocity is constant and equal to the average scalar velocity of an orbit: $v = v_m \approx 2 \cdot 10^6$ m/s (see related exercise in Series 1). We also know that the mass of this star is several orders of magnitude less than that of the black hole at the center of the orbit: $m \ll M$.

a) Using Newton's laws for the UCM, estimate the mass contained in the orbit of S2, in kg and in $M_\odot$.

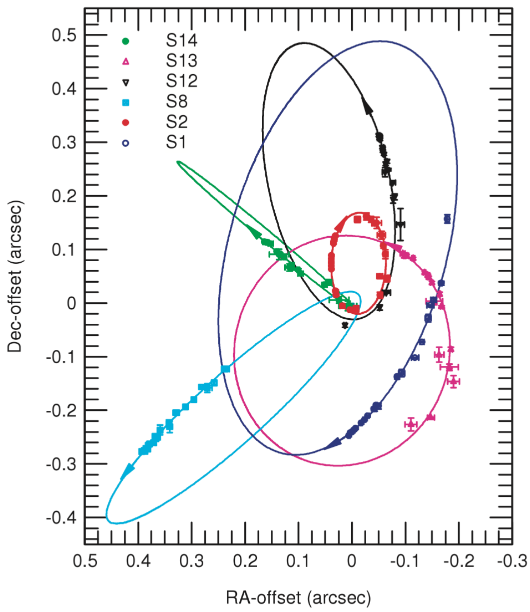
b) Calculate the Schwarzschild radius of that mass and compare it with the radius of S2's orbit.

c) We know that the perihelion of another star orbiting the supermassive black hole, S14 (also known as S0-16), is 45 AU. Why can we say that the mass contained in the orbit of these two stars is a black hole?

Astronomers were able to follow a complete rotation of S2 around Sgr A* from 1995 to 2010, allowing them to estimate the mass of the supermassive black hole in the core of our galaxy at about $4.2 \cdot 10^6$ $M_\odot$. As a result of this discovery Penrose et al. were awarded the Nobel Prize in Physics in 2020.
https://www.e-education.psu.edu/astro801/content/l8_p7.html.



Credit: NASA Chandra X-Ray Observatory e Penn State University.



Credit: Eisenhauer et al., 2005, ApJ 628, 246

Exercise 6: NCG1232 through a black hole

The next two images show the same galaxy, NCG1232, with a redshift of z =0.0053.

Credit: Application Copyright 2024, P. Lutus, http://arachnoid.com/relativity/index.html e https://it.wikipedia.org/wiki/NGC_1232

On the left, NCG1232 appears as we observe it now; on the right, it appears as we would observe it if a black hole of $4.3 \cdot 10^6$ $M_\odot$ were halfway between us (the observer, O) and NCG1232 (the source, S).

a) Answer the following questions :

1. What is the phenomenon that explains the formation of the image on the right?
2. Draw a diagram showing the positions of S, O and the black hole and the path of the light rays.
3. What is the name of the circle of light observed in the image on the right and what are the conditions for the formation of that circle?
4. How would the image on the right change if the black hole did not have perfect spherical symmetry?

b) Express the mass of the black hole in kg (SI units).

c) Calculate the Schwarzschild radius of the black hole and its average density.

d) From the redshift of NCG1232, determine the distance between us (O) and NCG1232 (S). Deduce the distance between O and the black hole and the distance between the black hole and S. Express all these distances in Mpc then in m.

e) Calculate the Einstein radius in the image on the right, expressing the result in arcseconds.

Exercise 7: Is our universe a black hole?

a) Write down the formula expressing the average density of a black hole as a function of its mass $\rho_{\mathrm{bh}}(M)$. Is it a function of M that is constant, increasing or decreasing?

b) From this formula determine the order of magnitude of the density of a black hole with a mass equal to:

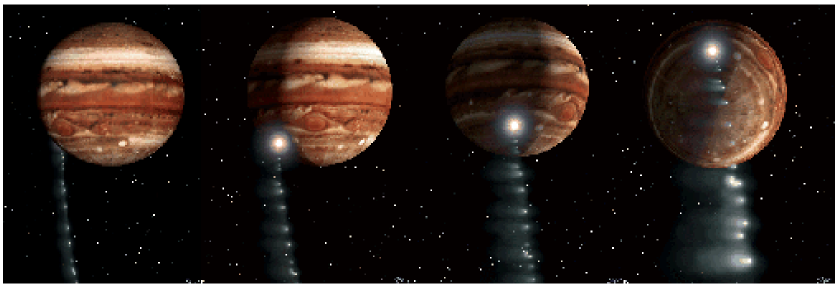
1. one ton: $\rho_{\mathrm{bh}}(1\ \mathrm{t})$;
2. that of the Earth: $\rho_{\mathrm{bh}}(M_{\mathrm{Earth}})$;
3. that of a large star (about 10 times the mass of the Sun): $\rho_{\mathrm{bh}}(10\ \mathrm{M}_\odot)$;
4. that of Gargantua: $\rho_{\mathrm{bh}}(10^8\ \mathrm{M}_\odot)$;
5. that of the core of our galaxy: SgrA*: $\rho_{\mathrm{bh}}(4,2 \cdot 10^6\ \mathrm{M}_\odot)$;
6. that of a sphere with radius equal to the Hubble radius $r_H = c/H_0$ and mean density equal to the critical density $\rho_c \approx 9 \cdot 10^{-27}\ \mathrm{kg/m^3}$.
   Hint: with the critical density, first find the mass contained in the Hubble radius: M_H, then calculate $\rho_{\mathrm{bh}}(M_H)$.

c) Calculate algebraically the result of question 6. (result in the form of a literal expression), knowing that the formula for the critical density is (Chapter 7)

$$\rho_c = \frac{3H_0^2}{8\pi G}\ .$$

Exercise 8: The comet Shoemaker-Levy 9

The collision of comet SL9 (Shoemaker-Levy 9) with Jupiter was observed by the Hubble telescope in July 1994. Prior to impact, the tidal forces experienced by the comet caused it to break up, creating a glittering trail of 21 fragments crashing into the red planet. Two years before the spectacular event in 1994, SL9's elliptical orbit had already reached a very close distance to Jupiter, some 40'000 km from its surface. Scientists assume that tidal forces fragmented the comet during this first close passage.



Composite photograph of SL9 fragments with Jupiter. Credit: https://solarsystem.nasa.gov/sl9/.

Video simulating the event https://youtu.be/wtYf GN-3yc?si=CUDXHtUV3Huugwe

a) Assuming SL9 to be a sphere of diameter $d = 5$ km and mass $m = 8$ billion tons, estimate the axial tidal force experienced by comet SL9 as it approached Jupiter in 1992.

b) What would be the same tidal force, in the same position, experienced by a person of mass $m = 60$ kg and height $h = 1.7$ m located perpendicular to the surface of the red planet?

c) What tidal force would a person at point (b) experience, at the same position, if Jupiter's mass were concentrated within its Schwarzschild radius? Give a qualitative answer, without performing calculations.

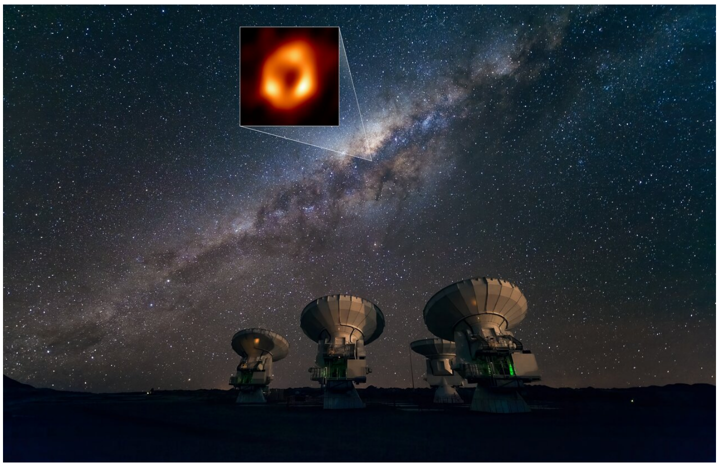
d) What would be the tidal force experienced by the same person as in (b) if Jupiter's mass were concentrated within its Schwarzschild radius and the person were at the position $r = 1.1$ km from its center?

Exercise 9: Tidal effect and temperature of SgrA*

a) Calculate the axial tidal force on a spacecraft of mass $m = 25.0$ tonnes and axial dimension $L = 18.0$ m venturing 1.50 km from the event horizon of the SgrA* black hole in the core of the Milky Way.

b) What is the temperature of the Sgr A* black hole in the core of the Milky Way?

c) Taking into account the average temperature of the current universe, why can't this black hole emit some Hawking radiation?



The Atacama Large Millimeter/submillimeter Array (ALMA) pointing at Sagittarius A*. The inset shows the radio image of Sagittarius A* taken by the Event Horizon Telescope (EHT) collaboration in 2021. Located in Chile's Atacama Desert, ALMA is the most sensitive observatory in the EHT network. Credit: ESO/José Francisco Salgado (josefrancisco.org), EHT Collaboration.

d) What would have to be the mass of a black hole for its temperature to be comparable to that of the present universe? What about its Schwarzschild radius?

Exercise 10: A tidal wave on the planet Miller

In the movie Interstellar, when the astronauts descend into the atmosphere of the planet Miller – orbiting the black hole Gargantua – their first vision reveals clouds and an Earth-like ocean. On landing, they discover that the ocean is only half a meter deep and, in the distance, they see what appears to be a vast mountain range. Later, they discover that this is a massive tidal wave, created by the powerful tidal forces exerted by Gargantua on the planet.

Miller is assumed to have the same mass and diameter as the Earth, and is located at a distance from the center of Gargantua equal to 10 times its Schwarzschild radius. Remember that the mass of the black hole is 10^8 $M_\odot$ (see ex. 4 of this series).

Left: view of Gargantua from the planet Miller. Right: scene of the tidal wave on the planet. Credit: The Interstellar movie.

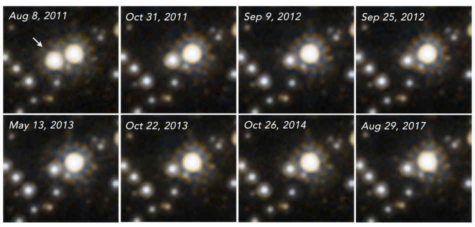
a) Estimate the axial tidal force created by Gargantua on planet Miller.

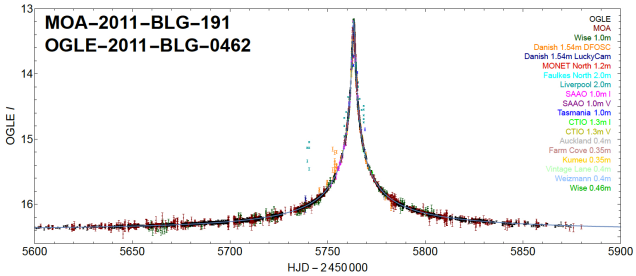
b) Compare the result of (a) with the tidal force of the Moon on the Earth: $F_{\text{tidal, Moon/Earth}} = 6.5 \cdot 10^{18}$ N, causing terrestrial tides (see Appendix B.3).

c) What would be the axial tidal force experienced by Miller if it were located at Gargantua's event horizon?

Exercise 11: The lonely black hole

The following images are from a study[8] published in July 2022. The first is a sequence of images of the same star (called "OGLE-2011" and indicated by the arrow in the first photo at top left) on different dates between 2011 and 2017. It can be seen that its brightness suddenly increased in August 2011 and then faded in the following months. The second is a relative graph of the star's brightness as a function of time over 300 days, centered on August 2011.





This increase in brightness is explained by the gravitational lensing effect created by the passage, between us and OGLE-2011 (the source), of a black hole called "MOA-

[8]Kailash C. Sahu *et al.*, *An Isolated Stellar-mass Black Hole Detected through Astrometric Microlensing*, https://arxiv.org/abs/2201.13296.

2011." The following table, taken from the same study, summarizes the data[9] of this event.

Table 6. Properties of the MOA-11-191/OGLE-11-462 Black Hole Lens

Property	Value	Sources & Notes[a]
Mass, $M_{\rm lens}$	$7.1 \pm 1.3\, M_\odot$	(1)
Distance, D_L	1.58 ± 0.18 kpc	(2)
Einstein ring radius, $\theta_{\rm E}$	5.18 ± 0.51 mas	(3)
Proper motion, (μ_α, μ_δ)	$(-4.36 \pm 0.22, +3.06 \pm 0.66)\,{\rm mas\,yr^{-1}}$	(4)
Galactic position, (X, Y, Z)	$(-4, -1580, -45)$ pc	(5)
Space velocities, (V, W)	$(+3, +40)\,{\rm km\,s^{-1}}$	(6)

Using the data in the table, answer the following questions:

a) What kind of gravitational lensing effect is this and why?

b) Is the lens of this system located in our galaxy? Justify your answer using the data in the table *without* doing calculations.

c) At this distance from us, can the expansion of the universe be neglected? Justify this statement.

d) How are black holes classified and to which of these categories does MOA-2011 belong?

e) Calculate (1) the Schwarzschild radius and (2) the average density of MOA-2011.

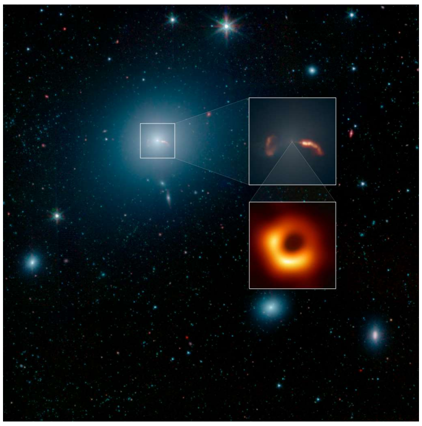
f) What would be the OM of the density of a black hole 1000 times more massive than MOA-2011?

g) Calculate the temperature of MOA-2011. Can it evaporate in our universe?

[9] mas = thousandth of an arcsecond

Exercise 12: Messier 87*

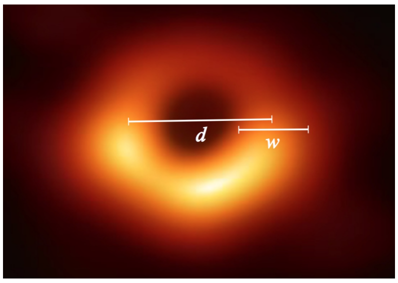
The image to the right shows an overview of Messier galaxy 87 (M87) and its neighbors. Two opposite jets of particles depart from the center of this galaxy, as can be seen in the first enlargement (top) in red and orange.
The second magnification (bottom) shows the image of the radio wave source M87*, containing the supermassive black hole that is the source of these jets. The radiation observed is that sent by the accretion disk of the rotating black hole. The axis of rotation of the black hole with respect to the observer (the Earth) is shown in the diagram below.



Credit: NASA, JPL-Caltech, IPAC, Event Horizon Telescope (EHT).

a) Explain what a supermassive black hole is: how does it differ from a stellar black hole?

b) Explain why the image of the accretion disk is a ring, even though its axis of rotation is not facing the observer. What phenomenon is this ? (If necessary, draw a diagram).



Credit: Jifeng, L. *et al.*, Nature, Vol. 575, 618–621 (2019).

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Table 1
Parameters of M87*

Parameter	Estimate
Ring diameter [a] d	42 ± 3 μas
Ring width [a]	<20 μas
Crescent contrast [b]	$>10:1$
Axial ratio [a]	$<4:3$
Orientation PA	150°–200° east of north
$\theta_g = GM/Dc^2$ [c]	3.8 ± 0.4 μas
$\alpha = d/\theta_g$ [d]	$11^{+0.5}_{-0.3}$
M[c]	$(6.5 \pm 0.7) \times 10^9\ M_\odot$
Parameter	**Prior Estimate**
D[e]	(16.8 ± 0.8) Mpc
M(stars) [e]	$6.2^{+1.1}_{-0.6} \times 10^9\ M_\odot$
M(gas) [e]	$3.5^{+0.9}_{-0.3} \times 10^9\ M_\odot$

Notes.
[a] Derived from the image domain.
[b] Derived from crescent model fitting.
[c] The mass and systematic errors are averages of the three methods (geometric models, GRMHD models, and image domain ring extraction).
[d] The exact value depends on the method used to extract d, which is reflected in the range given.
[e] Rederived from likelihood distributions (Paper VI).

For the following calculations, use the data below:

Black hole mass: $M = 6,5 \cdot 10^9$ M$_\odot$
Distance from Earth to M87*: $D = 16,8$ Mpc

Ring (angular) diameter: $d = 42 \cdot 10^{-6}$ $''$
Ring (angular) width: $w = 20 \cdot 10^{-6}$ $''$

c) Express in radians,

- Einstein's radius θ_E ;
- The angular radius of the "shadow" area within the ring, θ_{shadow}.

d) Calculate the Schwarzschild radius of the black hole in meters.

e) Is the event surface of the black hole within the "shadow" zone?

f) What is the density of the black hole? What substances have a density of the same order of magnitude?

g) Calculate the Hawking temperature of this black hole. Compare this temperature with the average temperature of the universe: can this black hole evaporate by emitting Hawking radiation?

Exercise 13: Evaporation times

When we restrict ourselves to classical (Newtonian or relativistic) physics, a black hole is an object that always attracts mass/energy and releases none. We have the impression that any physical information contained in what it engulfs is "lost", except for three (independent) physical quantities that uniquely define this object.

a) What are these three quantities?

b) Determine the order of magnitude of the evaporation time of a black hole of (1) $M_{\text{in}} = 1$ kg, (2) $M_{\text{in}} = M_{\text{Earth}}$ and (3) $M_{\text{in}} = 10\ \text{M}_\odot$.

c) What would have to be the initial mass of a black hole for it to evaporate in a period of time which is approximately the age of the universe, $t_{\text{evap}} \sim 10^{10}$ years?

Exercise 14: CERN's black hole

The LHC (Large Hadron Collider) at CERN can accelerate protons to a kinetic energy of $E_k = 13$ TeV.

a) Convert this energy into joules.

b) If it were possible to completely convert all this kinetic energy into mass energy, what would be the mass of the particle thus created?

c) What would be the Schwarzschild radius and density of a black hole of this mass?

d) What would be the estimated evaporation time of such a black hole?

Video of the simulation of a hypothetical stable black hole produced at CERN:
http://www.dailymotion.com/video/x7erd3_le-trou-noir-du-cern_tech

Exercise 15: Approaching a neutron star

Let's imagine that a person of height $h = 170$ cm and mass $m = 65$ kg is going to a distance of 5.0 km from the surface of a neutron star for a period of two days, in order to determine the rejuvenating (temporal) effect of gravity. For the neutron star use $M = 2,0\,\text{M}_\odot$ and $R = 10$ km.

a) What would be the axial tidal force on this person? Could they survive?

b) If this trip were feasible, how much younger would she be by the time she returns compared with her terrestrial counterparts?

c) Where would you have to go to get the same temporal effect, but without the wrenching effects of gravitational tides? Explain your answer by calculating the axial tidal force in the proposed location.

Exercise 16: Time travel

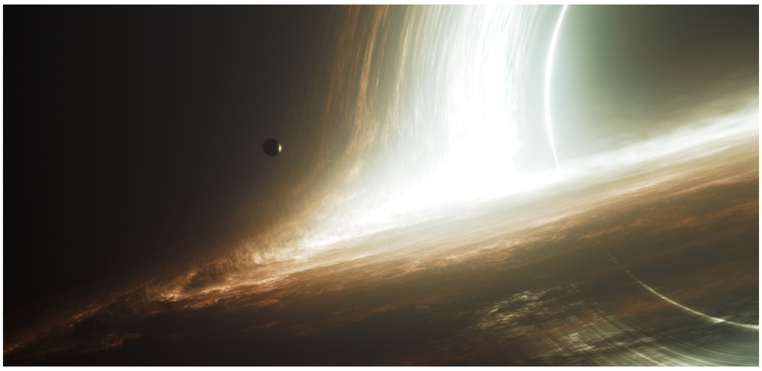
a) How far away from the surface of a 10 $M_\odot$ black hole would a 55-year old teacher have to be for one year for his 16-year-old students to be his age upon his return?

b) If, while traveling, he sends a picture' to his students at a frequency of 20 MHz, at what frequency will this signal be detected on Earth?

Exercise 17: Where is the planet Miller?

According to the information from the movie Interstellar, for every hour spent on the planet Miller, orbiting Gargantua, seven years pass in space far from the black hole (so also on Earth). Calculate the ratio between Miller's distance to the center of Gargantua and the black hole's Schwarzschild radius r/r_S. Is this result compatible with the images of the Miller planet?

For Gargantua's mass, use $M = 1.989\,100\,000 \cdot 10^{38}$ kg, or about 100 million $M_\odot$.



View of Gargantua and its planet Miller. Credit: Movie Interstellar.

Exercise 18: Cooper's return

In 2170, Pilot Cooper has returned to visit the black hole Gargantua ($M = 1.0 \cdot 10^8$ $M_\odot$). He does the round trip from Earth in a shuttle with constant velocity of $0.90\,c$ relative to the Solar System. According to the pilot's watch, the outward journey takes 1h 30', after which Cooper maintains a distance of $r = 3.0 \cdot 10^8$ km from the center of the black hole for 2h 45'. Finally, he returns to Earth at the same speed as the outward journey for 1h 30' (again according to his clock).

a) How much time has elapsed for an observer at rest on Earth during Cooper's absence?

b) What is the distance between Earth and Gargantua, in the Solar System reference system? Express the result in m and AU $= 1.5 \cdot 10^{11}$ m.

Exercise 19: GPS time lag

GPS satellites orbit the Earth at an altitude $h = 20.2 \cdot 10^6$ m above the Earth's surface. Their average speed is $3.89{\cdot}10^3$ m/s. We want to find out what difference there is between the time measured by the 'clock of an observer on the Earth's surface (considered at rest) and the time measured by a clock traveling on a GPS satellite.

Effect of special relativity:

a) Calculate, for each second, how much the GPS satellite clock lags behind the clock of an observer on Earth due to the satellite's speed.

Effect of general relativity:

b) Calculate, for each second, how much the clock of an observer on the Earth's surface lags behind an observer in space at an infinite distance from the Earth and from any other mass, due to gravitation.

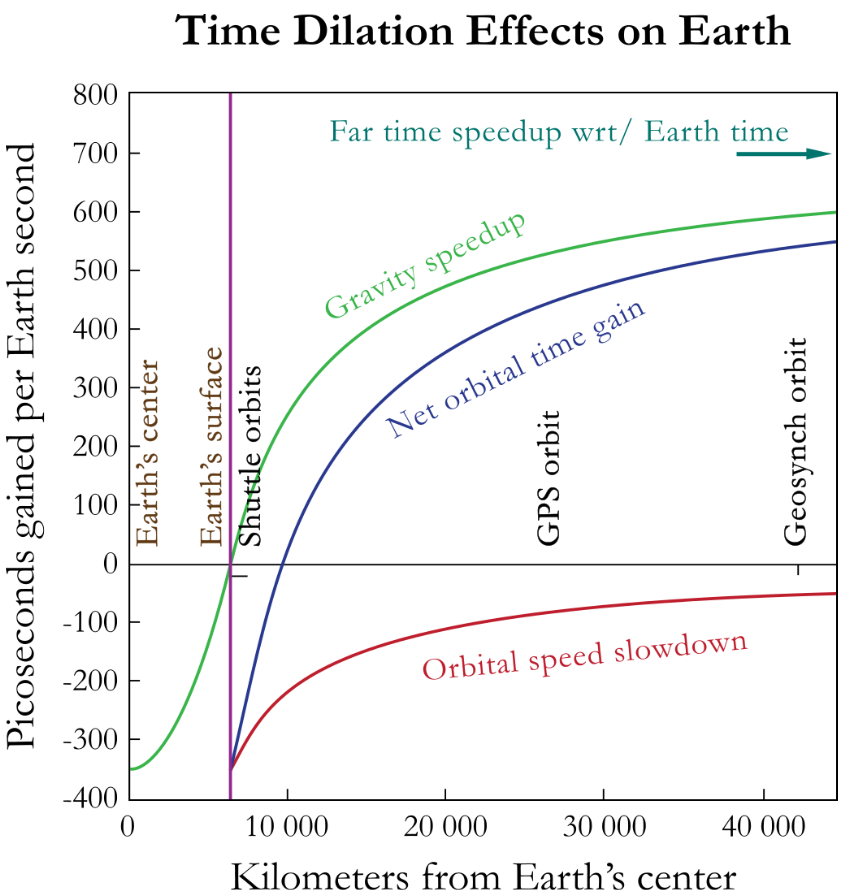
c) Repeat the same calculation as in the previous step, but for the delay of the GPS satellite clock with respect to an observer in space at an infinite distance from the Earth and from any other mass, due to gravitation.

d) From the previous two calculations, deduce the advance of the GPS satellite clock at each second relative to that of an observer on Earth, due to gravitation (general relativity).

Global relativistic effect:

e) Combining the effects of special and general relativity, what is the net effect on the time delay between the time measured by a GPS satellite and an observer on Earth ?

f) If the distance between the satellites and the Earth's surface h increases, do the effects of the contributions of (i) special relativity, (ii) general relativity, and (iii) total relativity increase or decrease? Compare the results with the following graph.



Credit: P. Fraundorf Derivative work, Spotsaurian

https://en.wikipedia.org/wiki/Error_analysis_for_the_Global_Positioning_System#/media/File:Orbit_times.svg

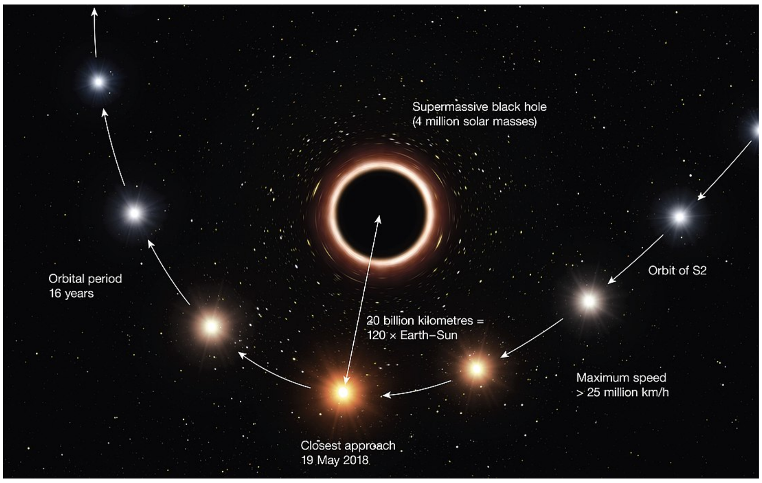
See also:

https://demonstrations.wolfram.com/RelativisticEffectsOnSatelliteClockAsSeenFromEarth/

Exercise 20: Elliptical orbits around Sgr A*

From the elliptical orbits of some stars around the center of the Milky Way, astronomers have been able to determine the presence of the black hole Sgr A*. In particular, the orbit of the star S2 (the reconstruction in the figure below shows part of its orbit) was followed throughout its 16.05-year period between 1995 and 2011. Its perihelion is at a distance of $r_p = 1.761 \cdot 10^{13}$ m and its eccentricity is $e = 0.885$.

a) From the data on the S2 orbit, determine the mass of Sgr A*. Express the result in kg and in solar masses.



Credit: ESO/M. Kornmesser.

b) What kind of black hole are we talking about?

c) Calculate the intensity of S2's velocity at perihelion and aphelion.

d) What is the Schwarzschild radius of Sgr A*? Express the result in m and solar radii ($R_\odot = 6.96 \cdot 10^8$ m).

e) Calculate the average density of Sgr A* and compare it with that of a stellar black hole.

f) What is the Hawking temperature of Sgr A*? Can it evaporate in our current universe?

g) If the surrounding temperature were low enough, how long would it take Sgr A* to evaporate completely (give the order of magnitude)? Compare the result with the order of magnitude of the age of the universe (14 billion years).

6 Cosmological equations

Exercise 1: Einstein's cosmological constant

a) Rewrite the first cosmological equation in a universe dominated by matter alone, clarifying the term containing the velocity of the scaling factor $\dot{a}^2(t)$. Explain why today, at time t_0, this equation does not admit a static solution. Recall that $a(t_0) = a_0 = 1$.

b) At the beginning of the 20th century, the idea that the universe is not static was difficult to accept. For this reason Einstein introduced a term into the first cosmological equation: the cosmological constant Λ_E. Rewrite the first cosmological equation for the matter-dominated universe, but with the addition of the term Λ_E. How much would Λ_E have to be worth today (at time t_0) for the universe to be static? Would it be a positive or negative value? Assume for the density of matter today (including dark matter) the value $\rho_{m0} = 10^{-27}$ kg/m^3.

c) Prove that – even if the universe were static today ($\dot{a}(t_0) = 0$) – it would be unstable: an infinitesimal velocity value ($\dot{a}(t_0) = \epsilon \ll 1$) would imply a non-zero acceleration $\dot{a}(t_0)$). What can be inferred from this?

To obtain the scale factor acceleration ä, derive the first cosmological equation with respect to time.

Exercise 2: In another universe

Let's imagine a universe filled only with a form of matter/energy whose density evolves as follows:

$$\rho = \frac{b}{a^5(t)}$$

where b is a constant. It is further assumed that this universe is flat.

a) Calculate the evolution of the scale factor as a function of time, $a(t)$ (remember that, at time $t = 0$, the scale factor is also equal to zero, $a = 0$).

b) For what values of t does this universe have a positive or negative expansion velocity? For what values of t is this expansion accelerating or decelerating?

c) Calculate the evolution of the Hubble parameter as a function of time, $H(t)$.

d) Calculate the evolution of density as a function of time $\rho(t)$, then express it as a function of a to check that the initial relationship ($\rho = b/a^5(t)$) is restored.

e) Use the second cosmological equation to find the equation of state of this form of energy and compare it with the equation of state of a gas of photons.

Exercise 3: The right distance

a) For the same source with redshift z_s, write the formulas

1. of the comoving distance D_0 ,
2. of the proper distance at the time of emission D_{em} and
3. of the crossing distance D_T .

b) Choose the correct answer and justify your choice:

The crossing distance D_T is.

- ☐ smaller than D_{em}.
- ☐ between D_{em} and D_0.
- ☐ larger than D_0.

c) Integrate each of the formulas written in (a), in the case $\Omega_m = 1$ and $\Omega_\Lambda = 0$ and find out what is the behavior (the limit) as a function of z_s when $z_s \to \infty$.

d) Are the approximations made in (c) reasonable for the observed high redshift sources, with $z_s \approx 10$?

e) Calculate the quantity $D_T(z_e)/c$ obtained in (c) for $z_s \to \infty$. What does it represent?

Exercise 4: Age of the universe (numerical method)

Use the last measurement results[10] for the parameters H_0, Ω_m and Ω_Λ to calculate from the crossing distance formula,

a) the age of the universe, and

b) how long ago the cosmological constant began to dominate the expansion (use $z_\Lambda = 0.3$).

Perform the integration numerically using, for example, Python, Octave, or Scilab.

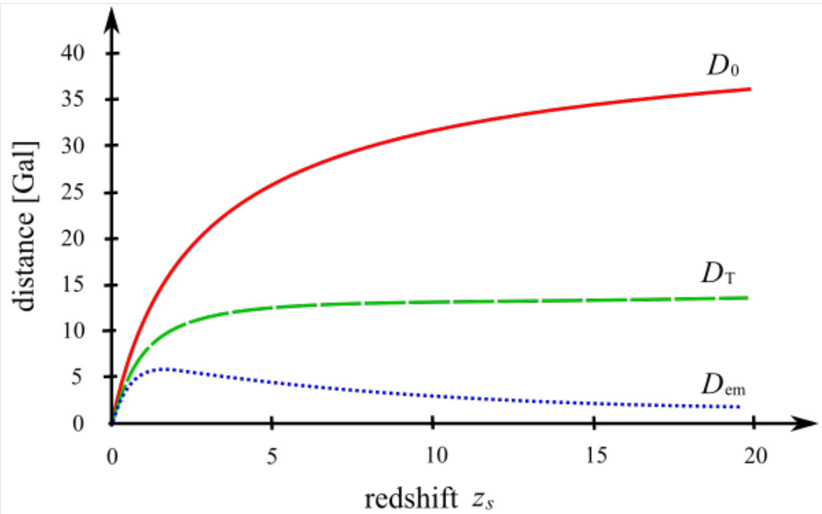
[10] https://en.wikipedia.org/wiki/Lambda-CDM_model

Exercise 5: Evolution of the emission proper distance

The graph below shows, as a function of redshift, D_0 (red solid line), D_T (green dashed line), and $D_{\rm em}$ (blue dashed line) obtained by numerical integration, with

$$H_0 = 70\,\frac{\rm km/s}{\rm Mpc} \quad , \quad \Omega_{\Lambda 0} = 0.7 \quad \text{and} \quad \Omega_{m0} = 0.3\ .$$

While for small z_s these three distances tend to be equal, for large z_s the difference between them becomes greater and greater, and they begin to differ before $z_s = 1$. As we have already seen in exercise 2, $D_{\rm em}$ tends to decrease when the redshift increases sufficiently.



a) What physical explanation can be given for this result?

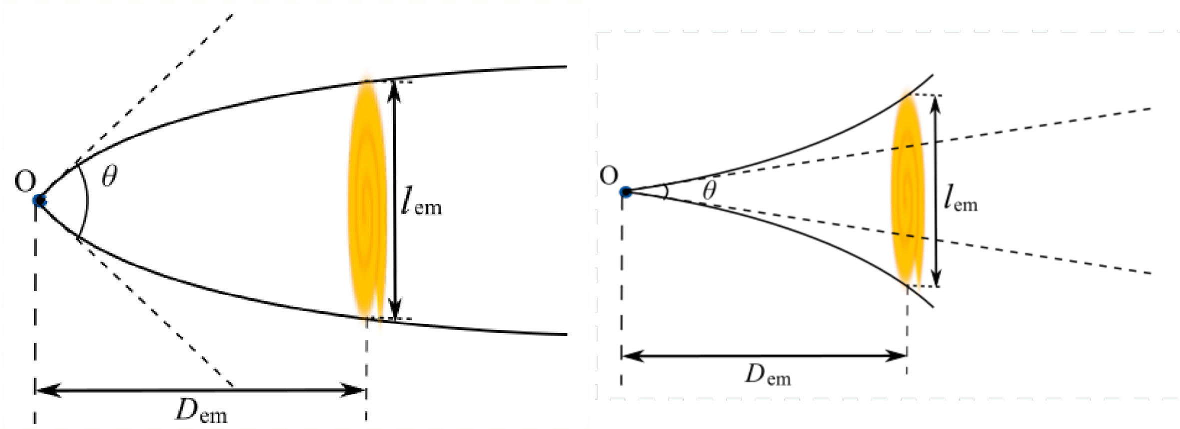
b) Assuming a matter-only universe ($\Omega_m = 1$) and using the results of exercise 2, determine at what redshift the proper distance at the time of emission would be maximum.

Exercise 6: Angular diameter distance $D_A(z_s)$

The *angular diameter distance* D_A is the apparent distance of a source deduced from the measurement of its angular diameter θ. It is used in measurements of the size of gravitational lenses or CMB fluctuations.

a) With what other cosmological distance does D_A coincide in an expanding flat universe? Write its integral formula as a function of z_s and cosmological parameters.

b) Complete the drawings below to explain why, in a non-zero curvature universe, D_A does not coincide with D_{em} and specify in which case D_A is larger or smaller than D_{em}.



Exercise 7: Luminosity distance in a matter Universe

The formula for the brightness distance of a source at redshift z_s is given by

$$D_L(z_s) = \frac{c\,(1+z_s)}{H_0} \cdot \int_0^z \frac{dz}{\sqrt{\Omega_m \cdot (1+z)^3 + \Omega_\Lambda}} \,.$$

a) Compare D_L with the other cosmological distances written in (a) of exercise 2: is it greater or lesser for a given z_s? How can this fact be explained?

b) Calculate analytically the integral of the luminosity distance when $\Omega_m = 1$ and $\Omega_\Lambda = 0$ and find out how it behaves as a function of z_s when $z_s \to \infty$.

c) Calculate analytically the same integral in the case where $\Omega_m = 0$ and $\Omega_\Lambda = 1$ and find how it behaves as a function of z_s when $z_s \to \infty$.

d) What is the effect of the cosmological constant Ω_Λ on the luminosity distance?

Exercise 8: Difference between cosmological distances

The expressions for the five cosmological distances as a function of redshift are integrals that can be solved analytically with all density parameters considered (exercise 9). However, there are programs that do this numerically for a source with a given redshift. For example on the following site:

https://ned.ipac.caltech.edu/help/cosmology_calc.html.

Using this program derive the five cosmological distances for sources placed at redshift 0.2, 2, 20, 200 respectively. Use the latest results[11] of measuring parameters H_0, Ω_m and Ω_Λ. What can we notice ?

z_s	D_L	D_0	D_T	$D_A = D_{\text{em}}$
0.20				
2.0				
20				
200				

Exercise 9: Dark energy (numerical method)

a) Use the latest results[2] in measuring parameters H_0, Ω_m and Ω_Λ and a suitable programming language (e.g., Python, Octave or Scilab) to draw the graph of $D_L(z_s)$ in Mpc. *Hint: make the graph in log10 scale.*

b) In the same graph, draw the *two* curves of the analytic functions of $D_L(z_s)$ for $\Omega_m = 1$ and for $\Omega_\Lambda = 1$ (questions (b) and (c) of exercise 6).

c) In the same graph, draw the points representing the D_L and z_S measurements for the 59 Ia supernovae obtained in 1998 by the Supernova Cosmology Project team[12] shown in the table below, where the magnitude measured for each supernova has been converted to a luminous distance.

d) What can you infer from this graph ?

[11] https://en.wikipedia.org/wiki/Lambda-CDM_model.
[12] http://supernova.lbl.gov/

n	D_L [Mpc]	z_s		n	D_L [Mpc]	z_s		n	D_L [Mpc]	z_s
0.. 0	3304	0.458		20	6166	0.828		40	2366	0.416
1	2168	0.354		21	3119	0.450		41	5297	0.830
2	1932	0.425		22	3062	0.430		42	129.4	0.030
3	1600	0.374		23	3565	0.580		43	243.2	0.05
4	2344	0.420		24	5675	0.763		44	119.1	0.026
5	2051	0.372		25	3090	0.526		45	351.6	0.075
6	2377	0.378		26	783.4	0.172		46	130.6	0.026
7	3119	0.453		27	4169	0.619		47	56.75	0.014
8	3357	0.465		28	5546	0.592		48	492.0	0.101
9	3999	0.498		29	3648	0.550		49	78.70	0.020
10	3266	0.655		30	883.1	0.180		50	155.6	0.036
11	2148	0.400		31	3664	0.374		51	241.0	0.045
12	3148	0.615		32	3034	0.472		52	198.6	0.043
13	2831	0.480		33	2366	0.430		53	95.94	0.018
14	2301	0.450		34	4227	0.657		54	326.6	0.079
15	2455	0.388		35	3963	0.612		55	520.0	0.088
16	3266	0.570		36	1706	0.320		56	322.10	0.063
17	3019	0.490		37	3597	0.579		57	335.7	0.071
18	2667	0.495		38	2667	0.450		58	233.3	0.052
19	3750	0.656		39	3006	0.581				

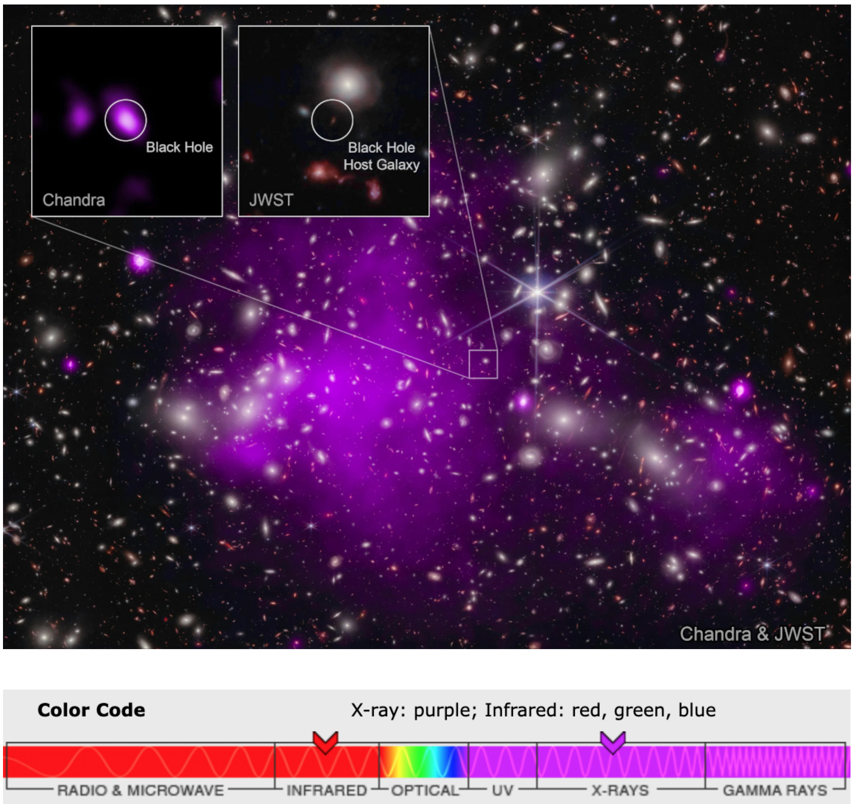
Source: http://arxiv.org/pdf/astro-ph/9812133v1.pdf

Exercise 10: Analytical expressions of cosmological distances

Analytically solve the integrals of cosmological distances as a function of redshift and density parameters, neglecting the curvature and radiation terms.

Exercise 11: The first JWST discoveries

The image below shows the galaxy cluster Abell 2744, whose redshift is $z_{\text{Abell}} = 0.308$. Thanks to the gravitational lensing effect of this cluster, the James Webb Space Telescope (JWST) was able to detect in 2023 the infrared radiation emitted by the most distant proto-galaxy (quasar) ever observed before, shown in the second insert of the figure: its name is UHZ1 (acronym for Ultra High redshift n. 1). The first insert shows the same proto-galaxy in X-rays, using the Chandra telescope. X-ray emission establishes the presence of a black hole with an estimated mass of 40 million solar masses within this source[13].



UHZ1 observed by the JWST and Chandra telescopes with the respective color coding. Credit: X rays: NASA/CXC/SAO/Ákos Bogdán; Infrared: NASA/ESA/CSA/STScI; Image processing: NASA/CXC/SAO/L. Frattare and K. Arcand; Color coding: https://chandra.harvard.edu/photo/2023/uhz1/.

[13]Credit: Goulding, Greene, Setton and Labbe, "UNCOVER: The Growth of the First Massive Black Holes from JWST/NIRSpec-Spectroscopic Redshift Confirmation of an X-Ray Luminous AGN at z = 10.1", *The Astrophysical Journal*, vol. 955 (2023).

11.1 The cluster Abell 2744

Answer the following questions, justifying each one.

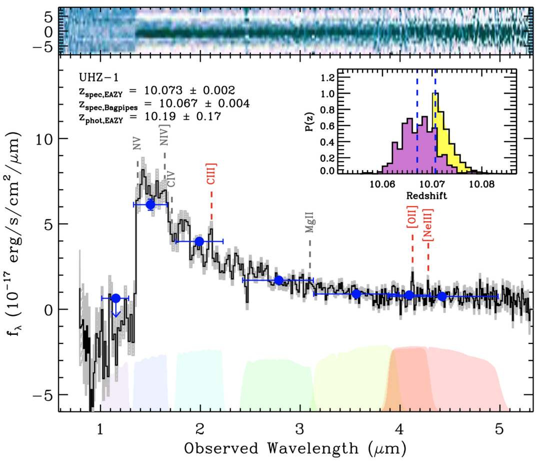
a) Determine the velocity at which the Abell 2744 cluster is moving relative to us. Is this velocity due to the cluster's motion in the surrounding space?

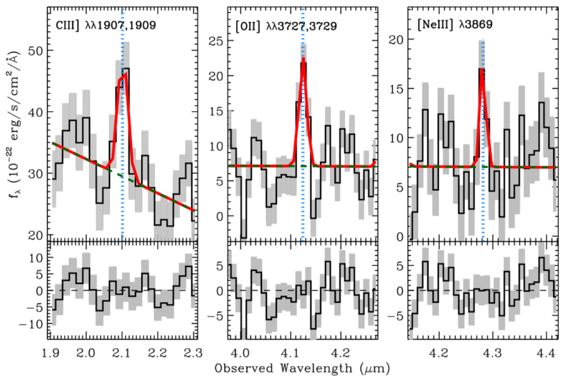
b) Determine the distance to the Abell 2744 cluster. Is it inside or outside the Hubble radius?

c) What kind of gravitational lensing effect(s) does this cluster create?

11.2 Distance of quasar UHZ1

The first graph below, on the left, shows the infrared radiation spectrum of UHZ1 (radiation intensity vs. observed wavelength λ_0), where the emission peaks of certain characteristic chemical elements can be observed. In particular, the inserts on the right illustrate the observed peaks of carbon (CIII), oxygen (OII) and Neon (NeIII). For each of these three elements, the respective wavelengths measured in the laboratory (λ) are given in Angstrom (1 Å $= 10^{-10}$ m).





Credit: Goulding et al. *The Astrophysical Journal* 955 https://arxiv.org/abs/2308.02750 (2023).

a) By reading the observed wavelength of the neon line (NeIII) from the graph, calculate the redshift of UHZ1.

b) Is this source inside or outside the Hubble radius? Justify and explain the answer with formulas.

c) Assuming a universe made up entirely of matter, estimate the crossing distance of UHZ1. Give the result in Mpc and al.

d) Still assuming a universe made up entirely of matter, what fraction of the age of the universe would it have taken UHZ1's light to reach us?

e) Still assuming a universe made up entirely of matter, what is the comoving distance of UHZ1? Give the answer in Mpc and compare it to D_T from the same source, calculated at point (c).

11.3 The black hole of UHZ1

a) How are black holes classified, and to which class does UHZ1 belong?

b) What are (1) the Schwarzschild radius and (2) the density of this black hole? Compare the density found with that of cotton candy ($\rho_{\text{cotton candy}} = 0.59$ kg/m^3).

c) Calculate the temperature of this black hole. If it had no accretion disk, could it lose mass through evaporation?

Explanation

The detection of this black hole, at such a density and at such a distance, was a key discovery in understanding the formation of supermassive black holes (SMBHs) in the nuclei of today's galaxies, a question debated for many years.

Indeed, the presence of supermassive black holes - i.e. "cold", relatively low-density black holes - even at the time of formation of the first structures in the universe[14] indicates that these nuclei were formed by the gravitational collapse of very large quantities of matter *before* the formation of the first stars, giving rise to giant black holes *without* initiating the nuclear combustion typical of stars. These first black holes, known as DCBHs (for "Direct Collapse Black Holes"), were the seeds that would later evolve into OBGs and then, through successive collisions, into today's galaxies, as shown in the following figure. The following video explains the significance of the discovery of Quasar UHZ1: https://www.youtube.com/watch?v=hxcUy-cBVcI.

[14]For this reason, these black holes are called OBGs, for "Outsize Black hole Galaxies"

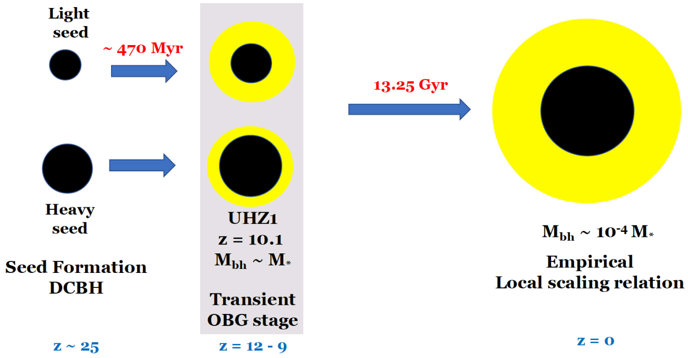
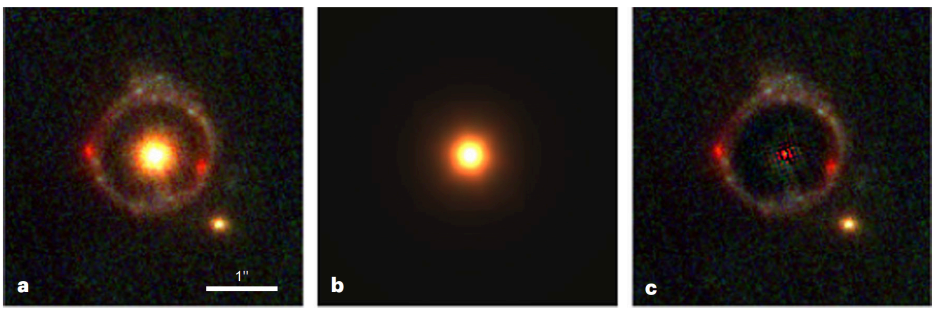


Diagram showing the evolution of supermassive black holes from the formation of the first structures ($z > 20$) to the present day ($z = 0$). The symbol $M_\star$ indicates the mass of the Galaxy, i.e. $\sim 10^{12}\ M_\odot$. Credit: Natarajan, Pacucci, Ricarte, Bogdan, Goulding and Cappelluti, "First Detection of an Over-Massive Black Hole Galaxy UHZ1: Evidence for Heavy Black Hole Seed Formation from Direct Collapse", *Synthical* (2023).

Exercise 12: Einstein ring as seen by JWST

The first Einstein Ring image taken by the James Webb telescope was named ER1 (for "Einstein Ring n. 1") and is shown in the figure below (image **a**): in image **b** we observe only the lens galaxy in the foreground, named ER1g (the g stands for "galaxy"), while image **c** represents only the near-perfect ring, named ER1r (the r stands for "ring").



Credit: Van Dokkum, P., Brammer, G., Wang, B. et al., *A massive compact quiescent galaxy at $z = 2$ with a complete Einstein ring in JWST imaging.* Nat Astron 8, 119–125 (2024). https://doi.org/10.1038/s41550-023-02103-9

The measured angular diameter of the ring is $1.54''$. Redshifts estimates of the ring-deformed source galaxy and the lens galaxy are $z_s = 2.98$ and $z_l = 1.94$ respectively.

From these measurements, scientists can calculate the mass of the lens galaxy. Since the redshifts of the two galaxies are significantly greater than unity, we can't use Hubble-Lemaître's law to calculate the distances involved in calculating the Einstein radius: in this case, it's necessary to use the angular distances which, in a flat universe, coincide with the emission distances of each galaxy.

a) Write down the formulae for the angular distances of the source and lens. Call them respectively

$$D_{A\,s} = D_s(t_{\mathrm{em}\,s}) = D_{SO} \quad \text{and} \quad D_{A\,l} = D_l(t_{\mathrm{em}\,l}) = D_{LO}\,.$$

Without doing any calculations, can you predict which is the biggest? Why?

b) Estimate the value of these distances by numerical calculation or by using one of the following numerical calculators: https://ned.ipac.caltech.edu/help/cosmology_calc.html

and the parameters $\Omega_m = 0,30$, $\Omega_\Lambda = 0,70$ and $H_0 = 70 \cdot 10^3$ m/(s · Mpc).

c) From the considerations made in (a), explain why the distance between the source and the lens to be considered is

$$D_{SL} = D_{SO} - \frac{1+z_l}{1+z_s} \cdot D_{LO}$$

then determine its value from the data and results of point (b).

d) Determine the mass of the ER1g lens. What type of material makes up this mass?

e) Calculate, in kpc, the radius around the lens galaxy delimiting the image of the ring, which we can consider to be the radius of ER1g.

f) Using 'initial mass formation" models, astrophysicists have estimated the stellar (luminous) mass of ER1g present within the radius of the ER1g galaxy to be $M_{\mathrm{lum}} = 1.1 \cdot 10^{11}$ $\mathrm{M}_\odot$. Compare the total mass of ER1g obtained at point (d) with its stellar mass, calculating their ratio.

Explanation

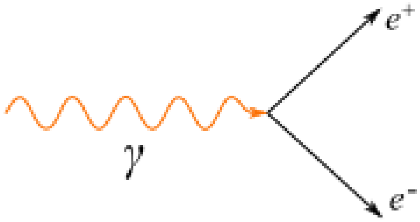
The observation of ER1 is of particular importance as it provides a measurement of the *total* mass (not just that of luminous matter) of a galaxy at relatively high redshift. The total mass of the lens is six times greater than the estimated stellar mass, which exceeds all predictions of previous astrophysical models,

7 Chronology of the Big Bang

Exercise 1: Pairs production

a) Determine the minimum energy E of a photon[15] so that it can be converted into:

- an electron/positron pair ($E_{\mathrm{e/po}}$);
- a proton/antiproton pair ($E_{\mathrm{pr/a}}$).



b) Are these values consistent with the corresponding energy ranges at the time of the respective particle annihilations given in the theory? Is it known why the particle/antiparticle pairs did not annihilate completely (today we observe protons and electrons but no antiparticles)?

c) Knowing that the relationship between the energy of a photon E and its associated wavelength λ is $E = hc/\lambda$, calculate the wavelength corresponding to the minimum energies to produce each pair of particles found in (a). In each case, determine the type of radiation involved (visible, infrared, UV, ...).

N.B. The wavelength associated with the mass energy of a particle, divided by 2π is called the De Broglie wavelength of the particle: $\lambdabar = \lambda/2\pi$.

d) Why, even if its energy would allow it, could a photon not be converted into a proton-neutron pair?

Exercise 2: Electrical potential energy

In Chapter 6 we obtained the formula for the *gravitational* potential energy of a system of two spherical masses, m_1 and m_2, whose centers are at a distance r.

a) Rewrite this formula. What is the sign of this energy?

[15]the particle associated with an electromagnetic wave

b) Using the analogy between the law of universal gravitation and Coulomb's law, write a formula similar to the one written in (a) that expresses the *electrical* potential energy of two charges q_1 and q_2, whose centers are at a distance r. What is the sign of this energy?

c) Use the formula in (b) and the data in Appendix A to calculate the electric potential energy of a proton and an electron in a hydrogen atom. Derive what this energy would be if, instead of having a proton-electron system, we had a proton-proton system, at the same distance?

d) Assuming that the electron follows a circular orbit around the nucleus[16] and, using Newton's second law, derive the formula for its orbital velocity as a function of e, r, m_e and k.

e) Derive the formulas for its kinetic energy and its mechanical energy.

Exercise 3: Recombination temperature

Imagine you have a plasma (a gas formed by charged particles) of protons and electrons.

a) Explain what it means for an ensemble of a large number of particles to be in thermal equilibrium, and why the primordial universe can be regarded as a plasma in thermal equilibrium.

b) Using the relationship between the temperature and the average energy of the particles of a plasma (Appendix E) and the result of exercise 2 (e), calculate the minimum temperature that a plasma must have for *all* its atoms to be ionized.

c) To what period in the thermal history of the universe does this temperature correspond?

Exercise 4: Strong interaction

The strong interaction (Appendix F) is one of nature's 4 fundamental interactions. It does not directly affect our daily lives because it becomes significant – and then dominant – only from a very small scale of distances, approximately a femtometer (1 fm = 10^{-15}m).

a) What objects do you know at this scale?

b) On what particles does the strong interaction act?

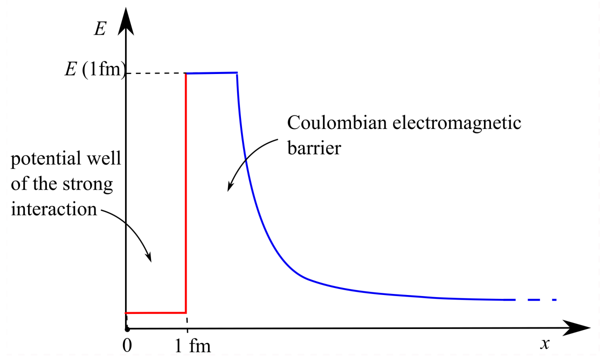
[16] This situation corresponds to Bohr's atomic model, an outdated model because electrons do not have precise velocities and positions in the atom. However, it gives a good estimate of the electron's mechanical energy

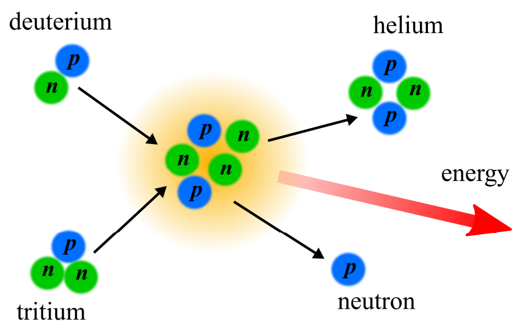
c) Would atoms be stable if it were not for the strong interaction?

d) Calculate the OM of the electrical potential energy of two protons approaching up to the distance where the strong interaction becomes more intense than the electrical one, i.e. about 1 fm.

The OM found in (d) is the minimum energy that protons would have to have to fuse to form a helium nucleus (this process is called *nuclear fusion*), without taking quantum effects into account. Nuclear fusion is a "difficult" process at ordinary temperatures, because very high energy must be provided to the protons to overcome the *coulombian barrier* of electrical potential energy, which is responsible for the repulsion between charges of the same sign.

But once this barrier is overcome, at a scale of 1 fm $= 10^{-15}$ m, the strong interaction between nuclei dominates and the proton falls into the attracting *potential well* of the strong interaction, which is responsible of the attraction between nuclei. This potential well is very deep and thus nuclear fusion releases enormous amounts of energy.





Exercise 5: Temperature associated with the coulombian barrier

Imagine that you have a plasma of protons (hydrogen nuclei). Using the relationship between the temperature and the average particle energy of a proton gas (Appendix E), calculate the minimum temperature of the primordial plasma for a significant number of its protons to pass the "Coulombian barrier" energy found in the previous exercise. What is the order of magnitude of this temperature?

Exercise 6: Nuclear fusion temperature and quantum tunnelling effect

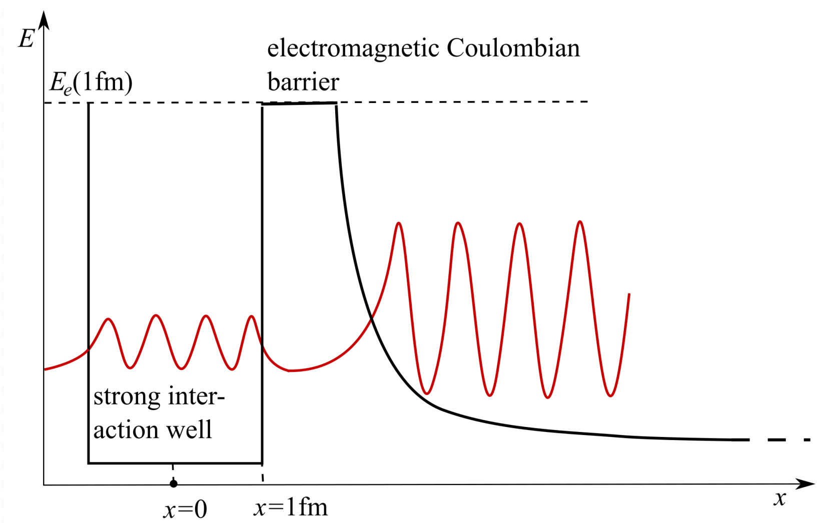
In fact, nuclear fusion is possible at temperatures several OM lower than those obtained in Exercise 5, where we considered protons as corpuscles that have to pass a potential (like a projectile that has to pass a wall). But in the subatomic microscopic world, it makes no sense to consider particles as point-like bodies whose position, velocity, acceleration and energy could theoretically be known at an arbitrarily precise instant. In fact, the concept of a solid "material body" exists only on a macroscopic scale.

a) Read Appendix G of the course carefully.

b) Knowing that the fusion temperature inside the Sun is 15 million Kelvin, determine the energy of the protons at this temperature (which is the energy at which the protons fuse).

c) Calculate the distance r_f at which, at this energy the protons can approach (the distance at which they begin to fuse).

d) Deduce the "size" of the "Coulombian barrier," that is, the length a in the drawing below. What can we conclude?



Exercise 7: Unification of gravitation and electromagnetism

By comparing the electromagnetic and gravitational forces between an electron and a proton in a hydrogen atom, but also between the Earth and the Moon (if we could take away all their electrons), we can see that the gravitational interaction is much weaker than the electromagnetic one (Series 1, exercise 6).

a) Now suppose we have two identical particles, with the same charge $q_1 = q_2 = q$ unknown, and the same mass $m_1 = m_2 = m$, also unknown. What must the ratio be m/q for the gravitational interaction between these two particles to be of the same intensity as the electromagnetic one?

b) What would have to be the mass m that the particles would have to have if their charge were equal to the fundamental charge ($q = e = 1.602 \cdot 10^{-19}$ C) for the gravitational and electromagnetic interaction between them to be of the same intensity?

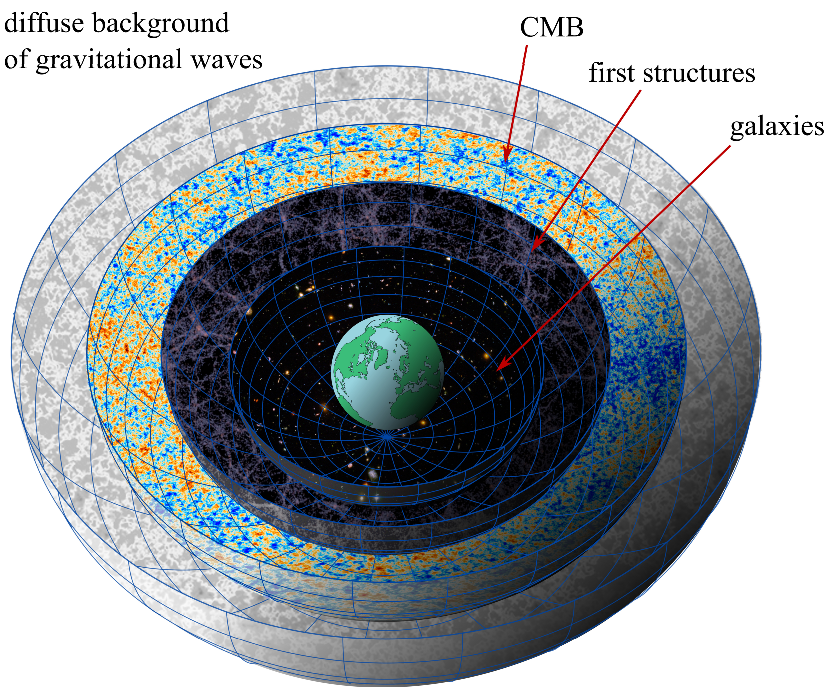
c) To what energy does this mass correspond? Give the answer in J, then in eV.

This means that if an elementary charge has an energy (kinetic and/or mass) of this order, then its gravitational interaction is comparable to the electromagnetic one, and therefore also to the nuclear one, since – at these energy scales – the electrical and nuclear interactions are already unified. If the energy of the particle is smaller, then gravitation is weaker than the other interactions. It is from this energy threshold that gravitation decouples from the primordial plasma.

d) Is the result obtained in (c) consistent with the value given in the theory for gravitational decoupling? Please comment on this result.

Bear in mind that at the time of electromagnetic decoupling of the primordial plasma (380'000 years after the Big Bang), the corresponding (electromagnetic) waves were able to escape and travel freely through space in the form of the Cosmic Microwave Background (CMB); at that time the universe became "transparent" to electromagnetic waves. Today, an analysis of this fossil radiation (a "photo of the baby universe") allows us to derive information from the universe at the time of (electromagnetic) decoupling, about 380'000 years after the Big Bang.

Similarly, at the moment of gravitational decoupling, gravitational waves corresponding to this interaction stopped interacting with the primordial plasma and were allowed to travel freely; the universe became transparent to gravitational radiation. Detection of these kinds of waves is difficult, because of their weakness (due to the nature of gravitation itself). But, for the same reason, a detection of the diffuse cosmological background of gravitational waves would allow us to establish a portrait of the universe at the time of gravitational decoupling, $t = 10^{-43}$ s after the Big Bang.



Exercise 8: True or false?

Justify each answer.

1. The energy density in the form of radiation was dominant for redshift $z < 10$.
2. In the Friedmann equation, currently the cosmological constant term dominates the expansion of the universe.
3. With expansion, the energy density of radiation decreases less rapidly than the density of matter.
4. The energy density of the vacuum remains constant despite the expansion of the universe.
5. Today's universe is filled with all the particles and antiparticles produced during the Big Bang.
6. Primordial nucleosynthesis took place after recombination: hydrogen and/or deuterium atoms fused to create atoms with a higher atomic number.
7. During primordial nucleosynthesis, all elements heavier than hydrogen that exist today were produced.

8. Most of the helium in the universe today was produced in stars.
9. The observed helium abundance is a confirmation of the Big Bang model.
10. All matter, baryonic or not, separated from the primordial plasma at the same time after recombination.
11. The *dark ages* are the period in the evolution of the universe when electromagnetic radiation is negligible.
12. The first stars formed after the Big Bang were much more massive than the stars we see today.
13. Reionization occurred after decoupling due to a short phase of contraction of the universe.

8 Gravitational Waves

Reminder: The following formulas relate the wavelength λ, period T, frequency f and propagation velocity v of a wave.

$$f = 1/T \qquad \lambda \cdot f = \lambda/T = v \ .$$

Notation: In this series, we call the angular velocity of a binary system

$$\omega = \omega_{\text{sys}} = \frac{2\pi}{T_{\text{sys}}} = 2\pi \cdot f_{\text{sys}}$$

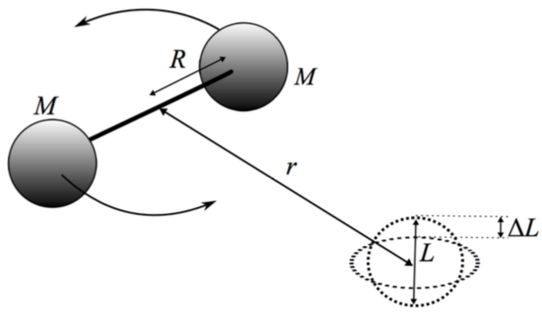
where T_{sys} is the period of rotation of the system and $f_{\text{sys}} = T_{\text{sys}}^{-1}$ its frequency, distinct

- from the period $T = T_{\text{gw}}$ and
- from the frequency $f = f_{\text{gw}} = T^{-1}$

the emitted gravitational wave.

Exercise 1: The dumbbell

A gravitational wave is generated by a binary system of two spherical bodies of the same mass M rotating at a distance $d = 2R$ from each other.
The wave propagates and reaches a ring of "test" matter of diameter L, located at a distance r from the center of rotation. The wave travels in the direction perpendicular to the plane of the ring and produces a relative strain on it, $\Delta L/L = h$ the formula for which is



$$h = \frac{\Delta L}{L} = \frac{2G}{c^4}\frac{M}{r}\omega^2 R^2 \qquad (1)$$

where ω is the angular velocity of rotation of the masses about their center[17]

a) Express equation (1) as a function of the Schwarzschild radius of M.

b) Explain why this quantity is always less than 1.

c) Calculate h for the waves generated by the rotation of a dumbbell composed of two masses of one ton each at a distance $d = 2$ m, and at a frequency $f = 1$ kHz, on a body at a distance $r = 10$ m.

[17] The position of the ring with respect to the axis of rotation of the system and the type of wave passing through it (transverse or longitudinal) have no influence on the magnitude of the relative deformation h.

Exercise 2: Relativistic rotational velocity: for which objects?

The formula for the amplitude h of a gravitational wave generated by a symmetric binary system can be written as the product of two dimensionless factors:

$$h = \frac{2GM}{c^2 r} \cdot \left(\frac{v}{c}\right)^2 = \frac{2GM}{c^2 r} \cdot \left(\frac{\omega R}{c}\right)^2 .$$

The first factor is of the same order (within a factor of 4) as the angle of deflection of a ray of light passing at a distance r from a mass M – we know that its numerical value is relatively small. The second factor is the ratio of the system's rotational speed to c.

Now consider a binary system consisting of a small mass m rotating around a large mass M ($m \ll M$) at a distance R.

a) Using Newton's second law applied to the uniform circular motion of m, prove that if the rotational velocity of m approaches that of light, then the radius of its circular trajectory R must approach half the Schwarzschild radius of the mass M:

$$v \approx c \quad \Rightarrow \quad R \approx \frac{r_S}{2} = \frac{GM}{c^2} .$$

b) Which objects have all their mass contained within r_S?

Exercise 3: An easily measurable strain?

Suppose there is a binary system rotating at a speed close to that of light ($v = \omega R \sim c$). This is possible only when the two masses have a distance comparable to their Schwarzschild radius ($R \sim r_S$, exercise 2).

a) In this case, what is the order of magnitude of the ratio M/r to have a relative deformation $h \sim 10^{-3}$, which is most easily measurable (a 1 meter ring would undergo a deformation of approximately 1 millimeter)?

b) What astrophysical objects exist that have this ratio?

Exercise 4: Frequency and generating mass

a) Explain why the frequency of a gravitational wave is twice the rotational frequency of the binary system that generates it:

$$f_{\text{gw}} = 2 \cdot f_{\text{sys}} \qquad \text{where} \qquad f_{\text{sys}} = 2\pi\, \omega_{\text{sys}} .$$

For the radiation to be as intense as possible, the masses of the system must rotate at a speed close to that of light ($v = \omega R \sim c$). To achieve this, they must rotate at a distance of the order of their Schwarzschild radius ($R \sim r_S$, see the result of exercise 3) and thus be very compact.

b) Using the relationship between the mass and Schwarzschild radius of a body and the result of (a), prove that the formula relating the OM of the wave frequency $f = f_{\text{gw}}$ and the mass of the components of the system M is

$$f \sim \frac{c^3}{2\pi\, G\, M} \, .$$

c) Determine the characteristic frequency of gravitational waves emitted by a 10 $\mathrm{M}_\odot$ system.

d) Perform the same calculation for a supermassive black hole system of 10^6 $\mathrm{M}_\odot$.

Exercise 5: Variation in the distance of a star

Among the most intense gravitational waves that could come from objects in our galaxy ($r \sim$ kpc) are those due to the final phase of the rotational motion of two black holes of one solar mass each, just before their collision. In this phase, the black holes orbit at a distance on the order of their Schwarzschild radius, thus at an orbital velocity close to that of light ($v = \omega R \sim c$).

a) Estimate what the relative strain h would be in this very optimistic case.

b) What would be the change ΔL in the distance between Earth and the nearest star, Proxima Centauri, caused by the passage of such a wave?

Exercise 6: Frequency and detector size

Each interferometer has a frequency band for which it is best suited for detecting gravitational waves, depending on its length L. In fact, if the period of oscillation of a wave is equal to or less than the time t taken by light to pass through the instrument $t = L/c$, when a photon passes through one arm of the interferometer, the distance between the two mirrors varies several times and thus different contributions to the ΔL strain of different signs overlap, altering the signal. A detector is considered sufficiently efficient when the duration t of the instrument crossing is at least 10 times smaller than the period T of the wave to be detected:

$$t = \frac{L}{c} \leq 0,1\, T \, .$$

a) Using the above considerations, write a formula that gives the maximum frequency a wave can have to be detectable by an instrument of size L.

b) What L is needed to detect a wave with a frequency between 10 Hz and 1 kHz? To what astrophysical objects do these frequency values correspond? Can gravitational wave detectors of this size be constructed?

c) The LISA space interferometer, which will be launched into heliocentric orbit in the coming decades, has "arms" of 5 million km. What frequencies will it be able to detect? To what astrophysical objects do these frequency values correspond?

Exercise 7: GW091415

The first historical gravitational wave detection was carried out on Sept. 14, 2015 (GW091415) from the two LIGO interferometers in the United States. It is the collision between two black holes of mass $M_1 = 29$ $M_\odot$ and $M_2 = 36$ $M_\odot$, at a distance $r = 400$ Mpc from Earth. The black hole obtained after the collision is 62 $M_\odot$.

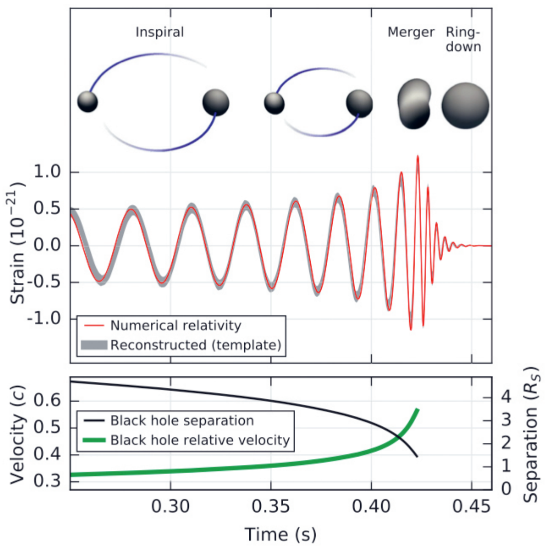
We were able to observe:

1. the signal of the last 8 periods of rotation of the two black holes, in a frequency band between 35 and 150 Hz;

2. the collision $f = 150$ Hz;

3. the "ringdown" phase, during which the final asymmetrically shaped black hole radiates waves to regain a spherical shape.

The total signal duration was 0.45 s. See Figure 2 of: https://www.nature.com/articles/s41586-019-1129-z.



Credit: B. P. Abbott et al., 2016, Observation of Gravitational Waves from a Binary Black Hole Merger, Physical Review Letters, 116, 061102 .

A simulation of this event can be seen at[18]:

https://www.youtube.com/watch?v=flvFpFUzEXY.

[18] More information is available on the LIGO collaboration website: https://www.ligo.org/detections/GW150914.php.

a) Determine the period of rotation of the system just before the collision.

b) From the distance r of the collision, determine the corresponding redshift. How many years ago did this collision occur?

c) What is the mass change of the system during the collision?

d) Assuming that all this mass was converted into gravitational radiant energy, what is the energy radiated in this collision? Give the answer to $\mathrm{M}_{\odot} \cdot \mathrm{c}^2$ and in J.

e) What is the average power radiated during the observed time? Are there any known phenomena with comparable power?

f) Calculate the yield of this collision: $\eta = \Delta E_{\text{radiated}}/E_{\text{tot}}$.

g) The most energetic nuclear fusion reaction is the following (production of a helium nucleus): $^{2}\mathrm{H} + {}^{3}\mathrm{He} \to {}^{4}\mathrm{He} + \text{proton} + 18.4$ MeV. Calculate its energy yield and compare it with that of the GW091415 collision. The mass of the proton is $m_p = 1.67 \cdot 10^{-27}$ kg $= 938$ MeV/c^2.